Portfolio Selection with Mental Accounts and Estimation Risk

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Abstract

In Das, Markowitz, Scheid, and Statman (2010), an investor divides his or her wealth among mental accounts with short selling being allowed. For each account, there is a unique goal and optimal portfolio. Our paper complements theirs by considering estimation risk. We theoretically characterize the existence and composition of optimal portfolios within accounts. Based on simulated and empirical data, there is a wide range of account goals for which such portfolios notably outperform those selected with the mean-variance model for plausible risk aversion coefficients. When short selling is disallowed, the outperformance still typically holds but to a considerably lesser extent.

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1. Introduction

Das, Markowitz, Scheid, and Statman (2010, DMSS) develop a model that incorporates aspects of both behavioral and mean-variance (hereafter 'MV') models. Like Shefrin and Statman (2000), DMSS consider an investor who divides his or her wealth among mental accounts (hereafter 'accounts') with motives such as retirement and bequest.[1] For each account, short selling is allowed and the optimal portfolio has maximum expected return subject to: (1) fully investing the wealth allocated to the account; and (2) the probability of the account's return being less than or equal to some threshold return (e.g., −20%) not exceeding some threshold probability (e.g., 1%).[2] Reflecting different account motives, the threshold return and threshold probability (hereafter 'thresholds') possibly vary across accounts. Nevertheless, optimal portfolios within accounts and the corresponding aggregate portfolio are on the MV frontier of Markowitz (1952). These portfolios also satisfy the safety-first criterion of Telser (1955).

When implementing a portfolio selection model in practice, an investor faces the risk of inaccurately estimating the optimization inputs (i.e., expected returns, variances, and covariances of available assets), which is referred to as *estimation risk*. While the literature has long recognized estimation risk in the MV model (see, e.g., Bawa, Brown, and Klein (1979)), it has yet to recognize estimation risk in the DMSS model. Our paper fills this gap.

We examine a model similar to the DMSS model, but an investor's optimal portfolio within a given account now has maximum *estimated* expected return subject to: (1) fully investing the wealth allocated to the account; and (2) the *estimated* probability of the account's return being less than or equal to the threshold return not exceeding the threshold probability.[3] Importantly,

[1] For an introduction to mental accounting, see Thaler (1985, 1999). Choi, Laibson, and Madrian (2009) provide empirical support for mental accounting in 401(k) plans. Also, the business press suggests that investors should divide their wealth into buckets dedicated to different goals so that they take the appropriate level of risk within each bucket; see, e.g., the article in *The Wall Street Journal*, October 5, 2012, pp. C9–C10. This article refers to two examples of buckets: (1) one dedicated to a car purchase in three years for which a relatively low level of risk would be appropriate; and (2) the other dedicated to tuition payments in 15 years for which a higher level of risk would be appropriate. Note that the meaning of 'buckets' in the article coincides with the meaning of 'accounts' in our paper.

[2] Formally, the optimal portfolio within a given account m solves $\max_{\boldsymbol{w}\in\mathbb{R}^N} \boldsymbol{w}'\boldsymbol{\mu}$ subject to $\boldsymbol{w}'\mathbf{1}_N = 1$ and $P[r_{\boldsymbol{w}} \leq H_m] \leq \alpha_m$. Here, $\boldsymbol{w}$ denotes a portfolio, N is the number of available assets, $\boldsymbol{\mu}$ is the $N \times 1$ vector of their expected returns, $\mathbf{1}_N$ is the $N \times 1$ unit vector, $P[\cdot]$ denotes probability, $r_{\boldsymbol{w}}$ is the random return on portfolio $\boldsymbol{w}$, H_m is the threshold return, and α_m is the threshold probability.

[3] Formally, the optimal portfolio within a given account m solves $\max_{\boldsymbol{w}\in\mathbb{R}^N} \boldsymbol{w}'\boldsymbol{\mu}^{\varepsilon}$ subject to $\boldsymbol{w}'\mathbf{1}_N = 1$ and $P^{\varepsilon}[r_{\boldsymbol{w}} \leq H_m] \leq$

we find that there is a wide range of thresholds for which the use of the DMSS model reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients. When short selling is allowed, we find that the DMSS model typically still reduces estimation risk (relative to the MV model), but to a lesser extent.

We begin by theoretically characterizing the existence and composition of optimal portfolios within accounts and the aggregate portfolio when short selling is allowed. First, we consider *fixed* thresholds that do not depend on the estimated optimization inputs (but possibly depend on the accounts). For example, the threshold return and probability for a given account might be -10% and 5%. The existence of the optimal portfolio within a given account depends on these thresholds and the estimated optimization inputs. If it exists, then it is on the estimated MV frontier. Hence, it would be selected by a hypothetical investor with an objective function defined over estimated expected return and variance for some risk aversion coefficient that also depends on the thresholds and inputs. Similar results hold for the aggregate portfolio.

Second, we consider *variable* thresholds that depend on the estimated optimization inputs. For example, the thresholds for a given account might be -7% and 5% for some inputs, and -9% and 4% for other inputs. Unlike fixed thresholds, variable thresholds can be set so that optimal portfolios within accounts and the aggregate portfolio: (1) exist regardless of the inputs; and (2) would be selected by hypothetical investors with risk aversion coefficients that do not depend on the inputs. As with fixed thresholds, the portfolios are on the estimated MV frontier.

Using simulated data, we then examine the existence and out-of-sample performance of the portfolios. In doing so, we consider eight assets: (a) Treasury bonds; (b) corporate bonds; and (c) the six size/book-to-market-based Fama-French equity portfolios.[4] We obtain two main findings. First, when fixed thresholds are used, optimal portfolios within accounts exist if and only if threshold

α_m. Here, $\boldsymbol{\mu}^\varepsilon$ is an estimate of $\boldsymbol{\mu}$ and $P^\varepsilon[\cdot]$ denotes estimated probability.

[4] Each of the 1000 simulations of estimated optimization inputs that we use is based on either 60 or 120 draws from a multivariate normal distribution with the mean vector and variance-covariance matrix associated with the asset monthly returns in 1978–2014. To assess the out-of-sample performance of the 1000 optimal portfolios within a given account (one portfolio for each simulation), we compute the average certainty equivalent return (CER) across simulations using the aforementioned mean vector and variance-covariance matrix; see Section 4.1.

returns are sufficiently small and threshold probabilities are sufficiently low. Second, there is a wide range of thresholds for which optimal portfolios within accounts have notably better out-of-sample performance than optimal portfolios in the MV model with plausible risk aversion coefficients.

We next use empirical data.[5] Compared to the findings based on simulated data, there are two main differences. First, optimal portfolios within accounts have lower out-of-sample performance. Second, the extent to which their out-of-sample performance exceeds that of optimal portfolios in the MV model is larger.

Using simulated and empirical data, we also examine the case where short selling is disallowed. Our findings differ from those in the case where it is allowed in three main respects. First, there is a larger set of fixed thresholds for which optimal portfolios within accounts exist. Second, regardless of whether fixed or variable thresholds are used, their out-of-sample performance is notably lower. Third, the extent to which their out-of-sample performance exceeds that of optimal portfolios in the MV model is considerably smaller.

Our paper complements DMSS along three dimensions. First, we theoretically characterize the existence and composition of optimal portfolios within accounts and the aggregate portfolio with fixed thresholds while recognizing estimation risk. Second, we theoretically characterize the set of variable thresholds for which the optimal portfolio within a given account: (i) exists regardless of the estimated optimization inputs; and (ii) would be selected by an investor with a risk aversion coefficient that does not depend on such inputs. Third, we examine the out-of-sample performance of optimal portfolios within accounts and the aggregate portfolio with fixed and variable thresholds. These dimensions are useful to investors who either have decided to implement the DMSS model (e.g., in setting thresholds and finding optimal portfolios) or are considering doing so (e.g., in assessing the relative out-of-sample performance of the DMSS and MV models).

[5] As with simulated data, we consider eight assets. In determining the estimated optimization inputs that correspond to the beginning of each year in the period 1983–2014, we use the previous 60 months of asset returns. Optimal portfolios within accounts and the aggregate portfolio are obtained by using such inputs and are assumed to be held during the forthcoming year. In assessing the out-of-sample performance of each of these portfolios, we compute its CER based on the monthly returns during this year and then compute its average CER across the 1983–2014 period. We proceed similarly when using 120 months to determine the inputs; see Section 5.1.

Also, our argument for justifying the use of the DMSS model complements theirs. Ours is that it reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients. Theirs relies on two assumptions: (1) investors specify account goals more precisely by stating thresholds instead of risk aversion coefficients; and (2) investors identify thresholds more precisely by stating them for portfolios within accounts instead of for the aggregate portfolio.

Our motivation for comparing the DMSS and MV models is threefold. First, since DMSS do so in the absence of estimation risk, it is natural to also do so in the presence of estimation risk. Second, in the case where short selling is allowed, while the literature notes the poor out-of-sample performance of the MV model when using plausible risk aversion coefficients, it is of interest to see if the DMSS model has notably better out-of-sample performance for a wide range of thresholds. Third, since the literature notes that disallowing short selling reduces estimation risk in the MV model, it is of interest to see if the DMSS model still typically outperforms the MV model when short selling is disallowed.

Examinations of estimation risk within the DMSS and MV models differ in four respects. First, while the DMSS investor has multiple accounts, the MV investor has a single account. Second, in determining optimal portfolios, the former investor uses different thresholds for different accounts whereas the latter uses a single risk aversion coefficient. Third, optimal portfolios in the DMSS model might not exist when using fixed thresholds (they exist when using variable ones), but those in the MV model always exist. Fourth, while the optimal portfolio within a given account correponds to the optimal portfolio in the MV model for some risk aversion coefficient that depends on the estimated optimization inputs (and on the thresholds), an MV investor utilizes a unique risk aversion coefficient that does not depend on such inputs.

Other recent papers also examine models with accounts in the absence of estimation risk. Alexander and Baptista (2011) consider an investor who delegates the management of his or her wealth to portfolio managers. Baptista (2012) and Jiang, Ma, and An (2012) consider investors who face, respectively, background risk (from sources such as labor income) and exchange rate risk. Our paper differs from theirs in three respects. First, the investor in our model faces estimation risk (but does not delegate the management of their wealth to portfolio managers nor face either background or exchange rate risk). Second, we consider *variable* thresholds. Third, we assess the *out-of-sample* performance of optimal portfolios within accounts.

A brief review of the literature on estimation risk in the MV model is in order. In terms of out-of-sample performance, Jorion (1986) finds that the use of shrinkage estimators for the optimization inputs is beneficial relative to the use of classical estimators. Frost and Savarino (1988) find that adding restrictions on portfolio weights reduces estimation risk. Best and Grauer (1991) show that optimal portfolios are very sensitive to the expected returns of available assets. Noting that such expected returns are difficult to estimate, Black and Litterman (1992) develop an approach in which they depend on both investor views and equilibrium expected returns. Chan, Karceski, and Lakonishok (1999) find that the estimation risk associated with the variance-covariance matrix is notable but smaller than that associated with the expected return vector.

Jagannathan and Ma (2003) show that disallowing short selling reduces estimation risk in the estimated minimum-variance portfolio even if the minimum-variance portfolio based on the 'true' variance-covariance matrix involves short positions. DeMiguel and Nogales (2009) show that the weights of portfolios based on certain robust estimators are more stable over time than those of the estimated minimum-variance portfolio, whereas the out-of-sample performance of the former portfolios is comparable to or slightly better than that of the latter. Kan and Zhou (2007) find that an optimal combination of (i) the risk-free asset, (ii) the estimated minimum-variance portfolio in the absence of this asset, and (iii) the estimated tangency portfolio has better out-of-sample performance than combinations of just (i) and (iii). Kan and Smith (2008) show that the estimated MV frontier is a notably biased estimator for the 'true' MV frontier and propose an alternative estimator that reduces this bias. DeMiguel, Garlappi, and Uppal (2009) find that the equally-weighted portfolio has better out-of-sample performance than optimal portfolios from the estimated MV model. Garlappi, Uppal, and Wang (2007) show that the optimal portfolio in a model where the expected return vector is contained in some set of expected return vectors and there is ambiguity aversion also has better out-of-sample performance. Michaud and Michaud (2008) discuss the limitations of the MV model that concern its implementation in practice. Our paper adds to this literature by finding that there is a wide range of thresholds for which the use of the DMSS model reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients.

We proceed as follows. Sections 2 and 3 theoretically characterize optimal portfolios within accounts and the aggregate portfolio with short selling allowed and, respectively, fixed and variable thresholds. Sections 4 and 5 assess their out-of-sample performance with, respectively, simulated and empirical data. Section 6 extends Sections 4 and 5 to the case where short selling is disallowed. Section 7 presents practical implications of our paper. Section 8 concludes. An online appendix contains our proofs.

2. The model

Let $N > 2$ be the number of available assets. We assume that their returns have a multivariate normal distribution.[6] Let $\boldsymbol{\mu}$ denote the $N \times 1$ vector of their expected returns. Its nth entry is asset n's expected return. We assume that $\boldsymbol{\mu}$ is not proportional to the $N \times 1$ unit vector, $\mathbf{1}_N$, so that at least two assets have different expected returns. Let $\boldsymbol{\Sigma}$ denote the $N \times N$ variance-covariance matrix for asset returns. Its entry in row n_1 and column n_2 is the covariance between the returns on assets n_1 and n_2. We assume that $\text{rank}(\boldsymbol{\Sigma}) = N$.[7]

[6] Several related papers also assume that asset returns have a multivariate normal distribution. DMSS and Jiang, Ma, and An (2012) do so in settings with multiple accounts where estimation risk is absent, whereas Kan and Zhou (2007) and DeMiguel, Garlappi, and Uppal (2009) do so in settings with a single account where estimation risk is present. Nevertheless, our results hold more generally in the case where asset returns are assumed to have a multivariate elliptical distribution (e.g., t distribution) with finite first and second moments. For an examination of optimal portfolios within accounts when asset returns are assumed to have non-elliptical distributions and estimation risk is absent, see Das and Statman (2013).

[7] The assumption that a risk-free asset is not available follows DMSS. Since they argue in favor of using their model, our model follows theirs as closely as possible (except for the issue of estimation risk). Further motivation for the aforementioned assumption can be found in, for example, Black (1972). Nevertheless, our results extend in a natural way to the case where a risk-free asset is available.

A portfolio is a $N \times 1$ vector $\boldsymbol{w}$ with $\boldsymbol{w}'\mathbf{1}_N = 1$. Its nth entry is asset n's weight. A positive (negative) weight represents a long (short) position. Let $r_{\boldsymbol{w}}$ denote portfolio $\boldsymbol{w}$'s random return. Its expected return and standard deviation are, respectively, $E[r_{\boldsymbol{w}}] \equiv \boldsymbol{w}'\boldsymbol{\mu}$ and $\sigma[r_{\boldsymbol{w}}] \equiv \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}$.

Let $\boldsymbol{\mu}^{\varepsilon}$ denote an estimate of $\boldsymbol{\mu}$. We assume that $\boldsymbol{\mu}^{\varepsilon}$ is not proportional to $\mathbf{1}_N$ so that at least two assets have different estimated expected returns. Similarly, let $\boldsymbol{\Sigma}^{\varepsilon}$ denote an estimate of $\boldsymbol{\Sigma}$. We assume that $\text{rank}(\boldsymbol{\Sigma}^{\varepsilon}) = N$. We refer to $\boldsymbol{\mu}^{\varepsilon}$ and $\boldsymbol{\Sigma}^{\varepsilon}$ as the estimated optimization inputs. For any given portfolio $\boldsymbol{w}$, we refer to $E^{\varepsilon}[r_{\boldsymbol{w}}] \equiv \boldsymbol{w}'\boldsymbol{\mu}^{\varepsilon}$ and $\sigma^{\varepsilon}[r_{\boldsymbol{w}}] \equiv \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}^{\varepsilon}\boldsymbol{w}}$ as its estimated expected return and standard deviation, respectively.

2.1. The investor's problem

Consider an investor who initially divides his or her wealth among a exogenously given number of accounts, denoted by $M \geq 2$. The $M \times 1$ vector of fractions of wealth in the accounts is exogenously given by $\boldsymbol{y} \in \mathbb{R}^M_{++}$ where $\boldsymbol{y}'\mathbf{1}_M = 1$ and $\mathbf{1}_M$ is the $M \times 1$ unit vector.[8] The investor then allocates the wealth within each account among the same set of assets. However, the portion of wealth within a given account that he or she allocates to any given asset possibly depends on the account.

Fixing estimated optimization inputs $\boldsymbol{\mu}^{\varepsilon}$ and $\boldsymbol{\Sigma}^{\varepsilon}$, the optimal portfolio within account m solves:

$$\max_{\boldsymbol{w} \in \mathbb{R}^N} \quad \boldsymbol{w}'\boldsymbol{\mu}^{\varepsilon} \tag{1}$$

$$s.t. \qquad \boldsymbol{w}'\mathbf{1}_N = 1 \tag{2}$$

$$P^{\varepsilon}[r_{\boldsymbol{w}} \leq H_m] \leq \alpha_m, \tag{3}$$

where $P^{\varepsilon}[\,\cdot\,]$ denotes estimated probability, $H_m \in \mathbb{R}$ is the threshold return, and $\alpha_m \in (0, 1/2)$ is the threshold probability.[9] Note that constraint (3) is tighter when either H_m is larger or α_m is

[8] The assumption that the number of accounts and the fraction of wealth in each account are exogenously given follows DMSS. As noted earlier, we follow them as closely as possible (except for the issue of estimation risk). Note that allowing the investor to endogenously determine the number of accounts and the fraction of wealth in each account might be inconsistent with the idea of having multiple accounts. Indeed, this idea breaks down if the investor ends up allocating 100% of his or her total wealth to a single account.

[9] Here, H_m and α_m are exogenous. Hence, given the estimated optimization inputs, the composition of the optimal portfolio within account m does not depend on the level of estimation risk (which depends on, for example, the number of months used

lower. The rest of Section 2 uses *fixed* thresholds that do not depend on the estimated optimization inputs. Section 3 uses *variable* thresholds that depend on such inputs.

Problem (1) subject to constraints (2) and (3) extends the problem that DMSS examine. First, the assumption that the investor maximizes the account's *estimated* expected return extends their assumption that he or she maximizes its '*true*' expected return. Second, the assumption that asset weights sum to one follows DMSS. Third, the assumption that the investor faces a constraint involving the *estimated* distribution of the account's return extends their assumption that he or she faces a constraint involving its '*true*' distribution.

Fix any portfolio $\boldsymbol{w}$. Its estimated Value-at-Risk (VaR) at confidence level $1-\alpha$ is:

$$V^{\varepsilon}[1-\alpha, r_{\boldsymbol{w}}] \equiv z_{\alpha}\sigma^{\varepsilon}[r_{\boldsymbol{w}}] - E^{\varepsilon}[r_{\boldsymbol{w}}], \tag{4}$$

where $z_{\alpha} \equiv -\Phi^{-1}(\alpha)$ and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function (cdf). Note that $z_{\alpha} > 0$ if $\alpha \in (0, 1/2)$. Also, an increase in the value of α reduces the size of z_{α}.

Portfolio $\boldsymbol{w}$ satisfies constraint (3) if and only if:

$$V^{\varepsilon}[1-\alpha_m, r_{\boldsymbol{w}}] \leq -H_m. \tag{5}$$

It follows from Eq. (4) that constraint (5) is equivalent to:

$$E^{\varepsilon}[r_{\boldsymbol{w}}] \geq H_m + z_{\alpha_m}\sigma^{\varepsilon}[r_{\boldsymbol{w}}]. \tag{6}$$

Hence, portfolios that lie on or above a line with intercept H_m and slope z_{α_m} in $(E^{\varepsilon}[r_{\boldsymbol{w}}], \sigma^{\varepsilon}[r_{\boldsymbol{w}}])$ space satisfy constraint (3), whereas those that lie below it do not; see Fig. 1A.

2.2. Optimal portfolios within accounts

We now proceed to characterize the existence and composition of optimal portfolios within accounts. Fixing the estimated optimization inputs, a portfolio is on the *estimated MV frontier* if

to determine these inputs). Sections 5 and 6 examine the case where H_m and α_m are endogenously set by maximizing the out-of-sample performance of this portfolio. In such a case, given the estimated optimization inputs, the composition of the portfolio depends on the level of estimation risk.

it minimizes estimated variance for some level of estimated expected return. For any level $E^\varepsilon \in \mathbb{R}$, the portfolio on this frontier is:

$$\boldsymbol{w}^\varepsilon_{E^\varepsilon} \equiv \phi^\varepsilon_{E^\varepsilon} \boldsymbol{w}^\varepsilon_0 + (1 - \phi^\varepsilon_{E^\varepsilon}) \boldsymbol{w}^\varepsilon_1. \tag{7}$$

Here, $\phi^\varepsilon_{E^\varepsilon} \equiv \frac{E^\varepsilon - B^\varepsilon/A^\varepsilon}{A^\varepsilon/C^\varepsilon - B^\varepsilon/A^\varepsilon}$ where $A^\varepsilon \equiv \mathbf{1}'_N(\boldsymbol{\Sigma}^\varepsilon)^{-1}\boldsymbol{\mu}^\varepsilon$, $B^\varepsilon \equiv (\boldsymbol{\mu}^\varepsilon)'(\boldsymbol{\Sigma}^\varepsilon)^{-1}\boldsymbol{\mu}^\varepsilon$, $C^\varepsilon \equiv \mathbf{1}'_N(\boldsymbol{\Sigma}^\varepsilon)^{-1}\mathbf{1}_N$, and $D^\varepsilon \equiv B^\varepsilon C^\varepsilon - (A^\varepsilon)^2$ are constants with C^ε and D^ε being positive. Portfolio $\boldsymbol{w}^\varepsilon_0 \equiv \frac{(\boldsymbol{\Sigma}^\varepsilon)^{-1}\mathbf{1}_N}{C^\varepsilon}$ has minimum estimated variance among all portfolios. Portfolio $\boldsymbol{w}^\varepsilon_1 \equiv \frac{(\boldsymbol{\Sigma}^\varepsilon)^{-1}\boldsymbol{\mu}^\varepsilon}{A^\varepsilon}$ lies in $(E^\varepsilon[r_{\boldsymbol{w}}], (\sigma^\varepsilon[r_{\boldsymbol{w}}])^2)$ space where a ray from the origin crosses the curve representing portfolios on the estimated MV frontier after passing through $\boldsymbol{w}^\varepsilon_0$. As the hyperbola in Fig. 1A illustrates, these portfolios can be represented in $(E^\varepsilon[r_{\boldsymbol{w}}], \sigma^\varepsilon[r_{\boldsymbol{w}}])$ space by using:

$$\sigma^\varepsilon[r_{\boldsymbol{w}}] = \sqrt{1/C^\varepsilon + \frac{(E^\varepsilon[r_{\boldsymbol{w}}] - A^\varepsilon/C^\varepsilon)^2}{D^\varepsilon/C^\varepsilon}}. \tag{8}$$

Hence, the asymptotic slope of the hyperbola is $\sqrt{D^\varepsilon/C^\varepsilon}$. Moreover, the estimated expected return of portfolio $\boldsymbol{w}^\varepsilon_0$ is $A^\varepsilon/C^\varepsilon$ and its estimated variance is $1/C^\varepsilon$.[10]

Let:

$$\alpha^\varepsilon \equiv \Phi(-\sqrt{D^\varepsilon/C^\varepsilon}). \tag{9}$$

Since $D^\varepsilon/C^\varepsilon > 0$, Eq. (9) implies that $\alpha^\varepsilon \in (0, 1/2)$. Also, the size of α^ε depends on the values of $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ (through terms C^ε and D^ε). For any $\alpha < \alpha^\varepsilon$, let:

$$H^\varepsilon_\alpha \equiv A^\varepsilon/C^\varepsilon - \sqrt{\frac{z^2_\alpha - D^\varepsilon/C^\varepsilon}{C^\varepsilon}}. \tag{10}$$

Using Eq. (10), the size of H^ε_α depends on the values of α as well as $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ (through terms A^ε, C^ε, and D^ε). If the confidence level is $1 - \alpha$, then the portfolio with minimum estimated VaR among all portfolios has an estimated VaR of $-H^\varepsilon_\alpha$; see the Appendix (Lemma 1).

Next, we characterize the existence and composition of optimal portfolios within accounts.

[10] The characterization of the estimated MV frontier in Eqs. (7) and (8) is similar to the characterization of the MV frontier in the absence of estimation risk; see, e.g., Huang and Litzenberger (1988, Ch. 3, hereafter 'HL'). Besides the issue of estimation risk, our theoretical results differ in three respects. First, we consider an investor with multiple accounts, whereas HL consider an investor with a single account. Second, while our investor has different goals for different accounts, HL's investor has a single goal. Third, for a given account, ours maximizes its estimated expected return subject to a constraint involving the estimated distribution of the account's return, whereas HL's maximizes an MV objective function.

Theorem 1. *Fix any account $m \in \{1, ..., M\}$. (i) If either (a) $\alpha_m \geq \alpha^\varepsilon$, or (b) $\alpha_m < \alpha^\varepsilon$ and $H_m > H^\varepsilon_{\alpha_m}$, then the optimal portfolio within account m does not exist. (ii) If $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$, then it exists and is given by:*

$$\boldsymbol{w}^\varepsilon_m \equiv \phi^\varepsilon_m \boldsymbol{w}^\varepsilon_0 + (1 - \phi^\varepsilon_m)\boldsymbol{w}^\varepsilon_1, \tag{11}$$

where $\phi^\varepsilon_m \equiv \frac{E^\varepsilon_m - B^\varepsilon/A^\varepsilon}{A^\varepsilon/C^\varepsilon - B^\varepsilon/A^\varepsilon}$. Here, its estimated expected return is:

$$E^\varepsilon_m \equiv A^\varepsilon/C^\varepsilon + \sqrt{(D^\varepsilon/C^\varepsilon)\left[(\sigma^\varepsilon_m)^2 - 1/C^\varepsilon\right]}, \tag{12}$$

and its estimated standard deviation is:

$$\sigma^\varepsilon_m \equiv \frac{z_{\alpha_m}(A^\varepsilon/C^\varepsilon - H_m) + \sqrt{(D^\varepsilon/C^\varepsilon)\left[(A^\varepsilon/C^\varepsilon - H_m)^2 - \left(z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon\right)/C^\varepsilon\right]}}{z^2_{\alpha_m} - D^\varepsilon/C^\varepsilon}. \tag{13}$$

Using Theorem 1, the existence of the optimal portfolio within account m ($\boldsymbol{w}^\varepsilon_m$) depends on the values of α_m and H_m as well as $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ (through terms α^ε and $H^\varepsilon_{\alpha_m}$). If $\alpha_m \geq \alpha^\varepsilon$, then it does not exist regardless of the size of H_m and $H^\varepsilon_{\alpha_m}$. As panels A and B of Fig. 1 show, its non-existence is due to the fact that estimated expected returns of portfolios satisfying constraint (6) do not have a finite upper bound. If $\alpha_m < \alpha^\varepsilon$, then its existence depends on the size of H_m and $H^\varepsilon_{\alpha_m}$. In the case where $H_m > H^\varepsilon_{\alpha_m}$, it does not exist. As panel C shows, its non-existence is due to the fact that no portfolio satisfies constraint (6). In the case where $H_m \leq H^\varepsilon_{\alpha_m}$, it exists. Panel D shows that it lies at the point p_m where the line is tangent to the curve when $H_m = H^\varepsilon_{\alpha_m}$. Similarly, panel E shows that it lies at the point p_m where the line crosses the top half of the curve when $H_m < H^\varepsilon_{\alpha_m}$.

Theorem 1 implies that the use of fixed thresholds requires that they are carefully set so that optimal portfolios within accounts exist. Fixing the estimated optimization inputs, if the optimal portfolio within a given account m does not exist with fixed thresholds α_m and H_m, then Theorem 1 is useful to reset the thresholds so that it does exist. Alternatively, as we show in Section 5, the use of variable thresholds guarantees that optimal portfolios within accounts exist.

When $\boldsymbol{w}_m^\varepsilon$ exists, it is on the estimated MV frontier; see Eqs. (7) and (11). Using Eqs. (12) and (13), the size of its estimated expected return E_m^ε and standard deviation σ_m^ε depends on the values of α_m, H_m, $\boldsymbol{\mu}^\varepsilon$, and $\boldsymbol{\Sigma}^\varepsilon$. While the use of a higher value of α_m loosens constraint (6) and thus increases their size, the use of a larger value of H_m tightens it and thus decreases their size.[11] The effect of $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ on the size of E_m^ε and σ_m^ε occurs through terms $A^\varepsilon/C^\varepsilon$, $1/C^\varepsilon$, and $D^\varepsilon/C^\varepsilon$. A larger value of $A^\varepsilon/C^\varepsilon$ shifts the hyperbola representing portfolios on the estimated MV frontier upward and thus increases their size. In contrast, a larger value of $1/C^\varepsilon$ shifts the hyperbola rightward and thus decreases their size. A larger value of $D^\varepsilon/C^\varepsilon$ shifts the top half of the hyperbola upward and thus increases their size.

Since $\boldsymbol{w}_m^\varepsilon$ is on the estimated MV frontier, it solves:

$$\max_{\boldsymbol{w}\in\mathbb{R}^N} \quad \boldsymbol{w}'\boldsymbol{\mu}^\varepsilon - \frac{\gamma_m^{i,\varepsilon}}{2}\boldsymbol{w}'\boldsymbol{\Sigma}^\varepsilon\boldsymbol{w} \tag{14}$$

$$s.t. \quad \boldsymbol{w}'\mathbf{1}_N = 1 \tag{15}$$

for some $\gamma_m^{i,\varepsilon} > 0$. We refer to $\gamma_m^{i,\varepsilon}$ as the *risk aversion coefficient implied by the optimal portfolio within account* m. Corollary 1 provides the value of $\gamma_m^{i,\varepsilon}$.

Corollary 1. *Fix any account $m \in \{1, ..., M\}$ with $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H_{\alpha_m}^\varepsilon$. The risk aversion coefficient implied by the optimal portfolio within account m is:*

$$\gamma_m^{i,\varepsilon} = \frac{D^\varepsilon/C^\varepsilon}{E_m^\varepsilon - A^\varepsilon/C^\varepsilon}. \tag{16}$$

Using Eqs. (12), (13), and (16), the size of $\gamma_m^{i,\varepsilon}$ depends on the values of α_m, H_m, $\boldsymbol{\mu}^\varepsilon$, and $\boldsymbol{\Sigma}^\varepsilon$. Since the use of a higher value of α_m increases the size of E_m^ε, it decreases that of $\gamma_m^{i,\varepsilon}$. In contrast, since the use of a larger value of H_m decreases the size of E_m^ε, it increases that of $\gamma_m^{i,\varepsilon}$. The effect of $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ on the size of $\gamma_m^{i,\varepsilon}$ occurs through terms $A^\varepsilon/C^\varepsilon$, $1/C^\varepsilon$, and $D^\varepsilon/C^\varepsilon$. Eqs. (12) and (16)

[11] In assessing the effect of an increase in a given term on the size of another term, we assume here (and hereafter) that the values of other terms remain unchanged.

imply that:

$$\gamma_m^{i,\varepsilon} = \sqrt{\frac{D^\varepsilon/C^\varepsilon}{(\sigma_m^\varepsilon)^2 - 1/C^\varepsilon}}. \tag{17}$$

Since a larger value of $A^\varepsilon/C^\varepsilon$ increases the size of σ_m^ε, it decreases that of $\gamma_m^{i,\varepsilon}$; see Eq. (17). In contrast, since a larger value of $1/C^\varepsilon$ decreases the size of E_m^ε, it increases that of $\gamma_m^{i,\varepsilon}$; see Eq. (16). A larger value of $D^\varepsilon/C^\varepsilon$ might either decrease, not affect, or increase the size of $\gamma_m^{i,\varepsilon}$; note that the right-hand side of Eq. (16) is affected by the value of $D^\varepsilon/C^\varepsilon$ in both the numerator and denominator (through term E_m^ε given by Eq. (12)).

2.3. Aggregate portfolio

If optimal portfolios within accounts exist, then aggregate portfolio $\boldsymbol{w}_a^\varepsilon \equiv \sum_{m=1}^M y_m \boldsymbol{w}_m^\varepsilon$ also exists. We characterize it next.

Theorem 2. *Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H_{\alpha_m}^\varepsilon$ for any account $m \in \{1, ..., M\}$. Then, the aggregate portfolio is given by:*

$$\boldsymbol{w}_a^\varepsilon = \phi_a^\varepsilon \boldsymbol{w}_0^\varepsilon + (1 - \phi_a^\varepsilon)\boldsymbol{w}_1^\varepsilon, \tag{18}$$

where $\phi_a^\varepsilon \equiv \sum_{m=1}^M y_m \phi_m^\varepsilon$. Its estimated expected return is:

$$E_a^\varepsilon \equiv \sum_{m=1}^{M} y_m E_m^\varepsilon, \tag{19}$$

and its estimated standard deviation is:

$$\sigma_a^\varepsilon \equiv \sqrt{1/C^\varepsilon + \frac{(E_a^\varepsilon - A^\varepsilon/C^\varepsilon)^2}{D^\varepsilon/C^\varepsilon}}. \tag{20}$$

When aggregate portfolio $\boldsymbol{w}_a^\varepsilon$ exists, it is on the estimated MV frontier; see Eqs. (7) and (18). Using Eqs. (19) and (20), the size of its estimated expected return E_a^ε and standard deviation σ_a^ε depends on the fractions of wealth in the accounts, the thresholds (which affect E_m^ε for $m = 1, ..., M$), and the estimated optimization inputs.

Since $\boldsymbol{w}_a^{\varepsilon}$ is on the estimated MV frontier, it solves:

$$\max_{\boldsymbol{w}\in\mathbb{R}^N} \quad \boldsymbol{w}'\boldsymbol{\mu}^{\varepsilon} - \frac{\gamma_a^{i,\varepsilon}}{2}\boldsymbol{w}'\boldsymbol{\Sigma}^{\varepsilon}\boldsymbol{w} \tag{21}$$

$$s.t. \quad \boldsymbol{w}'\mathbf{1}_N = 1 \tag{22}$$

for some $\gamma_a^{i,\varepsilon} > 0$. We refer to $\gamma_a^{i,\varepsilon}$ as the *risk aversion coefficient implied by the aggregate portfolio.* Corollary 2 provides the value of $\gamma_a^{i,\varepsilon}$.

Corollary 2. *Suppose that $\alpha_m < \alpha^{\varepsilon}$ and $H_m \leq H_{\alpha_m}^{\varepsilon}$ for any $m \in \{1, ..., M\}$. Then, the risk aversion coefficient implied by the aggregate portfolio is:*

$$\gamma_a^{i,\varepsilon} = \frac{D^{\varepsilon}/C^{\varepsilon}}{E_a^{\varepsilon} - A^{\varepsilon}/C^{\varepsilon}}. \tag{23}$$

Using Eqs. (12), (13), (19), and (23), the size of $\gamma_a^{i,\varepsilon}$ depends on the fractions of wealth in the accounts, the thresholds for the accounts, and the estimated optimization inputs.

3. Variable thresholds

We now use variable thresholds, which depend on the estimated optimization inputs as noted earlier. In doing so, we focus on thresholds for which optimal portfolios within accounts exist regardless of these inputs and imply risk aversion coefficients that also do not depend on the inputs. Our motivation is twofold. First, when using fixed thresholds, optimal portfolios within accounts might not exist (see Theorem 1). Second, when using variable thresholds, optimal portfolios within accounts coincide with optimal portfolios in the MV model for risk aversion coefficients that do not depend on the inputs.[12] The latter portfolios can thus be found by using such thresholds.

[12] Note that when fixed thresholds are used, they are primitives for characterizing the behavior of the investor within the accounts. In contrast, when variable thresholds are used, a possible interpretation is that the primitives for characterizing this behavior are risk aversion coefficients that do not depend on the estimated optimization inputs. While our motivation for using variable thresholds is not based on this interpretation, Kan and Zhou (2007) and DeMiguel, Garlappi, and Uppal (2009) develop settings with a single account and estimation risk where optimal portfolios are obtained by using an objective function with a risk aversion coefficient that does not depend on these inputs. Besides the use of the two types of thresholds (i.e., fixed and variable) having different implications for the existence of optimal portfolios within accounts and the size of their implied risk aversion coefficients as discussed earlier, these two types of thresholds also differ in terms of complexity. By design, variable thresholds are more complex than fixed thresholds in that the former need to be computed whereas the latter are given. Since each type of thresholds is of interest on its own, we present results for both types.

3.1. Optimal portfolios within accounts

For any $\gamma^i > 0$, let:

$$\alpha^{\varepsilon,\gamma^i} \equiv \Phi\left(-\sqrt{\frac{D^\varepsilon + (\gamma^i)^2}{C^\varepsilon}}\right). \tag{24}$$

Since $C^\varepsilon > 0$, $D^\varepsilon > 0$, and $\gamma^i > 0$, Eq. (24) implies that $\alpha^{\varepsilon,\gamma^i} \in (0, 1/2)$. Also, the size of $\alpha^{\varepsilon,\gamma^i}$ depends on the values of $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ (through terms C^ε and D^ε) as well as γ^i.

Next, we characterize optimal portfolios within accounts.

Theorem 3. *Fix any account $m \in \{1, ..., M\}$ and any constant $\gamma^i_m > 0$. Suppose that the thresholds are given by $\widetilde{\alpha}_m$ and $\widetilde{H}_m$, where:*

$$\widetilde{\alpha}_m \leq \alpha^{\varepsilon,\gamma^i_m} \tag{25}$$

and

$$\widetilde{H}_m = \frac{A^\varepsilon}{C^\varepsilon} + \frac{D^\varepsilon}{\gamma^i_m C^\varepsilon} - z_{\widetilde{\alpha}_m}\sqrt{\frac{1}{C^\varepsilon} + \frac{D^\varepsilon}{(\gamma^i_m)^2 C^\varepsilon}}.^{13} \tag{26}$$

Then, the optimal portfolio within account m exists and is given by:

$$\widetilde{\boldsymbol{w}}^\varepsilon_m \equiv \widetilde{\phi}^\varepsilon_m \boldsymbol{w}^\varepsilon_0 + (1 - \widetilde{\phi}^\varepsilon_m)\boldsymbol{w}^\varepsilon_1, \tag{27}$$

where $\widetilde{\phi}^\varepsilon_m \equiv \frac{\widetilde{E}^\varepsilon_m - B^\varepsilon/A^\varepsilon}{A^\varepsilon/C^\varepsilon - B^\varepsilon/A^\varepsilon}$. Here, its estimated expected return and standard deviation are, respectively:

$$\widetilde{E}^\varepsilon_m \equiv \frac{A^\varepsilon}{C^\varepsilon} + \frac{D^\varepsilon}{\gamma^i_m C^\varepsilon} \tag{28}$$

and:

$$\widetilde{\sigma}^\varepsilon_m \equiv \sqrt{\frac{1}{C^\varepsilon} + \frac{D^\varepsilon}{(\gamma^i_m)^2 C^\varepsilon}}, \tag{29}$$

where γ^i_m is its implied risk aversion coefficient.[14]

[13] The use of the tilde ('$\sim$') in $\widetilde{\alpha}_m$ indicates that $\widetilde{\alpha}_m$ is *variable*. While $\widetilde{\alpha}_m$ depends on the values of $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ as well as γ^i_m, for brevity we write '$\widetilde{\alpha}_m$' instead of '$\widetilde{\alpha}^{\varepsilon,\gamma^i_m}_m$.' The tilde is similarly used in $\widetilde{H}_m$.

[14] While there are always variable thresholds $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ for which the optimal portfolio within account m exists regardless of the estimated optimization inputs and has a given implied risk aversion coefficient γ^i_m that does not depend on such inputs, $\widetilde{\alpha}_m$ cannot exceed $\alpha^{\varepsilon,\gamma^i_m}$; see Eq. (25). First, assume that $\widetilde{\alpha}_m \geq \alpha^\varepsilon$. Note that the optimal portfolio within account m does not exist; see Theorem 1. Second, assume that $\alpha^{\varepsilon,\gamma^i_m} < \widetilde{\alpha}_m < \alpha^\varepsilon$. While the optimal portfolio within account m lies on the estimated MV frontier, it cannot lie below the portfolio with minimum estimated VaR at confidence level $1 - \widetilde{\alpha}_m$. However, the portfolio that solves problem (14) subject to constraint (15) with $\gamma^{i,\varepsilon}_m = \gamma^i_m$ lies below the portfolio with minimum estimated VaR at confidence level $1 - \widetilde{\alpha}_m$.

Theorem 3 implies that there is a set of variable thresholds for which the optimal portfolio within a given account m: (i) exists regardless of the estimated optimization inputs; and (ii) has a given implied risk aversion coefficient γ_m^i that does not depend on these inputs. For fixed inputs, this set has infinitely many elements, but each of these elements leads to the same optimal portfolio with account m. Intuitively, Fig. 1E shows that this portfolio lies where the line crosses the top half of the hyperbola; see point p_m. However, there are infinitely many lines with 'appropriate' slopes and vertical intercepts (corresponding to 'appropriate' threshold probabilities and returns, respectively) that also cross it at p_m.

Note that the optimal portfolio within account m, $\widetilde{\boldsymbol{w}}_m^\varepsilon$, is on the estimated MV frontier; see Eqs. (7) and (27). Using Eqs. (28) and (29), the size of its estimated expected return $\widetilde{E}_m^\varepsilon$ and standard deviation $\widetilde{\sigma}_m^\varepsilon$ depends on the values of γ_m^i, $\boldsymbol{\mu}^\varepsilon$, and $\boldsymbol{\Sigma}^\varepsilon$. A larger value of γ_m^i decreases their size.[15] The effect of $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ on the size of $\widetilde{E}_m^\varepsilon$ occurs through terms $A^\varepsilon/C^\varepsilon$ and $D^\varepsilon/C^\varepsilon$. A larger value of either term increases its size. Similarly, the effect of $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ on the size of $\widetilde{\sigma}_m^\varepsilon$ occurs through terms $1/C^\varepsilon$ and $D^\varepsilon/C^\varepsilon$. A larger value of either term increases its size.

3.2. Aggregate portfolio

Next, we characterize the composition of the aggregate portfolio.

Theorem 4. *For any account $m \in \{1, ..., M\}$, suppose that $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ satisfy, respectively, Eqs. (25) and (26) where $\gamma_m^i > 0$. Then, the aggregate portfolio is:*

$$\widetilde{\boldsymbol{w}}_a^\varepsilon \equiv \widetilde{\phi}_a^\varepsilon \boldsymbol{w}_0^\varepsilon + (1 - \widetilde{\phi}_a^\varepsilon)\boldsymbol{w}_1^\varepsilon, \tag{30}$$

where $\widetilde{\phi}_a^\varepsilon \equiv \sum_{m=1}^M y_m \widetilde{\phi}_m^\varepsilon$. Its estimated expected return and standard deviation are:

$$\widetilde{E}_a^\varepsilon \equiv \frac{A^\varepsilon}{C^\varepsilon} + \frac{D^\varepsilon}{\gamma_a^i C^\varepsilon} \tag{31}$$

[15] In deriving this partial equilibrium result, we assume that $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ remain unchanged (see footnote 11). An examination of a general equilibrium model with accounts and estimation risk is left for future research.

and:

$$\widetilde{\sigma}_a^{\varepsilon} \equiv \sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{\left(\gamma_a^i\right)^2 C^{\varepsilon}}}, \tag{32}$$

respectively, where $\gamma_a^i \equiv \left(\sum_{m=1}^{M} y_m/\gamma_m^i\right)^{-1}$ *is its implied risk aversion coefficient.*

Note that aggregate portfolio $\widetilde{\boldsymbol{w}}_a^{\varepsilon}$ is on the estimated MV frontier; see Eqs. (7) and (30). Using Eqs. (31) and (32), the size of its estimated expected return $\widetilde{E}_a^{\varepsilon}$ and standard deviation $\widetilde{\sigma}_a^{\varepsilon}$ depends on the fractions of wealth in the accounts, the implied risk aversion coefficients of optimal portfolios within accounts, and the estimated optimization inputs.

4. Simulated data

In this section, we use simulated data to examine the existence and out-of-sample performance of optimal portfolios within accounts and the aggregate portfolio. As we explain shortly, the use of simulated data allows us to consider the case where the first two moments of the distribution of asset returns are assumed to be constant over time (Section 5 considers the case where they possibly vary over time).

4.1. Methodology

Our methodology takes eight steps. In step 1, we specify the available assets: (a) Treasury bonds; (b) corporate bonds; and (c) the six size/book-to-market-based Fama-French equity portfolios.[16] Returns on Treasury and corporate bonds are extracted from Bloomberg by using the corresponding Bank of America Merrill Lynch indices. Returns on the Fama-French equity portfolios are obtained from Kenneth French's website. Table 1 presents summary statistics on the monthly asset returns during 1978–2014.

In step 2, we specify the accounts. We consider three accounts ($m = 1, 2, 3$).[17] Also, we assume that the fractions of wealth in these accounts are given by $(y_1, y_2, y_3) = (40\%, 30\%, 30\%)$.[18]

[16] In illustrating their theoretical results, DMSS use three assets with one of them being analogous to a bond and the other two being analogous to stocks. Similarly, we use assets that involve bonds and stocks.

[17] In illustrating their theoretical results, DMSS also consider three accounts.

[18] The results for aggregate portfolios are similar when using other reasonable values for the fractions of wealth in the accounts. Note that the results for optimal portfolios within accounts are not affected by the values of such fractions.

In step 3, we obtain 60 draws from a multivariate normal distribution with: (1) a mean vector that corresponds to the average returns in the first row of Table 1; and (2) a variance-covariance matrix that corresponds to the standard deviations and the correlation coefficients in, respectively, the second and last eight rows. In step 4, we use the 60 draws to obtain simulation 1 of the estimated optimization inputs. In step 5, we use such inputs to examine the existence of optimal portfolios within accounts. When they exist, we find their composition and implied risk aversion coefficients as well as the composition of the aggregate portfolio and its implied risk aversion coefficient. In step 6, we repeat steps 3–5 for simulations $2, ..., 1000$ of the estimated optimization inputs. In step 7, we compute the *average CER* of the 1000 optimal portfolios within each account (one portfolio for each simulation). Let $\boldsymbol{w}^{\varepsilon}_{m,s}$ denote the optimal portfolio within account m in simulation s for $m = 1, 2, 3$ and $s = 1, ..., 1000$. For any account $m \in \{1, 2, 3\}$ and any risk aversion coefficient $\gamma > 0$, the average CER of portfolios $\{\boldsymbol{w}^{\varepsilon}_{m,s}\}_{s=1}^{1000}$ is $\overline{CER}^{\varepsilon}_{m,\gamma} \equiv \frac{\sum_{s=1}^{1000} E[r_{\boldsymbol{w}^{\varepsilon}_{m,s}}] - \frac{\gamma}{2}(\sigma[r_{\boldsymbol{w}^{\varepsilon}_{m,s}}])^2}{1000}$.[19] Here, $E[r_{\boldsymbol{w}^{\varepsilon}_{m,s}}]$ and $\sigma[r_{\boldsymbol{w}^{\varepsilon}_{m,s}}]$ are obtained by using the mean vector and variance-covariance matrix noted in step 3. Hence, the first two moments of the distribution of asset returns are assumed to be constant across simulations. Similarly, we compute the average CER of the 1000 aggregate portfolios (again, one portfolio for each simulation). In step 8, we repeat steps 3–7 by using 120 (instead of 60) draws.[20]

4.2. Optimal portfolios within accounts

This section considers optimal portfolios within accounts.

4.2.1 Fixed thresholds

We begin by examining the existence of optimal portfolios within accounts. Fig. 2 reports the fraction of simulations for which the optimal portfolio within a given account m exists as a function of threshold probability α_m and threshold return H_m. Panels A and B use, respectively, 60 and

[19] Similar results are obtained when using Sharpe ratios (instead of CERs) to assess out-of-sample performance. Our reported results use CERs for both brevity and consistency with the fact that there is no risk-free asset in the DMSS model (we obtain monthly returns on Treasury Bills from Kenneth French's website to calculate these ratios). In a setting with a single account and estimation risk, Kan and Zhou (2007) argue in favor of using CERs instead of Sharpe ratios to assess out-of-sample performance.

[20] We also use 180 draws. The results mainly differ from those reported for the cases of 60 and 120 draws in that optimal portfolios within accounts and the aggregate portfolio have larger average CERs.

120 draws. Four results can be seen. First, the fraction is 0% (i.e., there is no simulation for which the portfolio exists) if either: (i) α_m is sufficiently low and H_m is sufficiently large; or (ii) α_m is sufficiently high. Second, the fraction is strictly between 0% and 100% (i.e., the portfolio exists in some but not all simulations) if either: (a) α_m is sufficiently low and H_m is within some relatively small range; or (b) α_m is within some relatively large range. Third, the fraction is 100% (i.e., the portfolio exists in all simulations) if α_m is sufficiently low and H_m is sufficiently small. Fourth, the size of the set of thresholds for which the fraction is 100% increases in the number of draws. Hence, thresholds should be carefully set so that optimal portfolios within accounts exist, particularly when using a relatively small number of observations to determine the estimated optimization inputs.

Next, we assess the out-of-sample performance of optimal portfolios within accounts. Table 2 shows their average CERs.[21] In computing the average CERs for accounts 1, 2, and 3, we use risk aversion coefficients of, respectively, 4, 3, and 1.[22] In the first and second set of three columns to the right of the 'Account' column, the number of draws is, respectively, 60 and 120. Panel A uses fixed thresholds. In the first three rows, they are exogenously given by $(\alpha_1, \alpha_2, \alpha_3) = (1\%, 5\%, 10\%)$ and $(H_1, H_2, H_3) = (-5\%, -8\%, -10\%)$. Note that average CERs are all positive.[23] Also, they increase in the number of draws (due to the estimated optimization inputs becoming more precise).[24]

We now examine the relation between the average CERs and the values of thresholds. Using threshold probabilities given by $(\alpha_1, \alpha_2, \alpha_3) = (1\%, 5\%, 10\%)$, panels A and B of Fig. 3 show the

[21] Average CERs are well-defined since we compute tem only for thresholds such that optimal portfolios within accounts exist in all simulations. In general, however, when fixed thresholds are used, there is a positive probability of obtaining a simulation for which the optimal portfolio within a given account does not exist. Theorem 1 says that the optimal portfolio within account m does not exist if either: (i) $\alpha_m \geq \alpha^{\varepsilon}$ (since estimated expected returns of portfolios satisfying constraint (6) do not have a finite upper bound); or (ii) $\alpha_m < \alpha^{\varepsilon}$ and $H_m > H_{\alpha_m^{\varepsilon}}$ (since no portfolio satisfies constraint (6)). However, the probability of non-existence is zero if: (1) asset weights are bounded; and (2) for each simulation where $\alpha_m < \alpha^{\varepsilon}$ and $H_m > H_{\alpha_m^{\varepsilon}}$, the threshold return increases to a value not exceeding $H_{\alpha_m^{\varepsilon}}$. While (1) guarantees that estimated expected returns of portfolios satisfying constraint (6) have a finite upper bound, (2) guarantees that there is a portfolio satisfying constraint (6). When implementing the DMSS model in practice, conditions (1) and (2) are realistic.

[22] These coefficients are reasonable in the context of related work. In the numerical example of DMSS, optimal portfolios within accounts have implied risk aversion coefficients of 3.80, 2.71, and 0.88. Moreover, Kan and Zhou (2007) and DeMiguel, Garlappi, and Uppal (2009) consider risk aversion coefficients of, respectively, 3 and 1.

[23] In order to reduce estimation risk within the MV model, some researchers suggest the use of either the estimated minimum-variance portfolio (see, e.g., Chan, Karceski, and Lakonishok (1999) and Jagannathan and Ma (2003)) or the equally-weighted portfolio (see, e.g., DeMiguel, Garlappi, and Uppal (2009)). While a detailed horse race between the performance of such portfolios and that of optimal portfolios within accounts is beyond the scope of our paper, we find that the latter portfolios typically outperform the former with some exceptions in the case where short selling is disallowed and empirical data are used.

[24] This finding does not necessarily suggest the use of a sample with the largest possible size to determine the estimated optimization inputs. For example, Sharpe (2000, p. 179) notes that there is an increasing likelihood that the underlying probability distribution is unstable as the sample size increases, resulting in increasingly unreliable estimates.

average CERs for various threshold returns with, respectively, 60 and 120 draws. In each panel, the solid, dashed, and dotted lines represent accounts 1, 2, and 3, respectively. In both panels, average CERs initially increase in the threshold return, but then decrease. Similarly, using threshold returns given by $(H_1, H_2, H_3) = (-5\%, -8\%, -10\%)$, panels C and D show the average CERs for various threshold probabilities with, respectively, 60 and 120 draws. In both panels, average CERs initially increase or are relatively constant in the threshold probability, but then sharply decrease.

In the middle three rows of Table 2A, threshold probabilities are exogenous as in the first three rows whereas threshold returns are endogenously set by maximizing the average CERs of optimal portfolios within accounts. In setting them, we compute the average CER for each element in an appropriate grid of threshold returns and then identify the element that leads to the largest average CER.[25] Note that endogenous threshold returns decrease in the number of draws. In the case of accounts 1, 2, and 3, the increases in average CERs arising from using endogenous threshold returns instead of exogenous ones (along with exogenous threshold probabilities) are, respectively: (a) 0.13%, 0.02%, and 0.15% with 60 draws; and (b) 0.33%, 0.03%, and 0.90% with 120 draws.

In the last three rows, threshold returns are exogenous as in the first three rows whereas threshold probabilities are endogenously set by maximizing the average CERs of optimal portfolios within accounts. In setting them, we compute the average CER for each element in an appropriate grid of threshold probabilities and then identify the element that leads to the largest average CER. Note that endogenous threshold probabilities increase in the number of draws. In the case of accounts 1, 2, and 3, the increases in average CERs arising from using endogenous threshold probabilities instead of exogenous ones (along with exogenous threshold returns) are respectively: (a) 0.12%,

[25] Recognizing that the values of the thresholds that maximize average CERs generally depend on the 'true' optimization inputs and such inputs are not precisely known, these values cannot be exactly determined in practice. However, assuming that the 'true' optimization inputs in practice are relatively 'close' to the 'true' optimization inputs in our paper, it is of interest to examine such values for three reasons. First, the values of the thresholds that maximize average CERs in our paper provide some indication on the kinds of values that maximize average CERs in practice. Second, the use of the former values allow us to obtain a rough upper bound on the benefit arising from considering estimation risk in the DMSS model (by comparing average CERs with endogeneous and exogenous thresholds). Third, the use of the aforementioned values also allow us to obtain a rough upper bound on the benefit arising from using the DMSS model instead of the MV model with plausible risk aversion coefficients. An important point of our paper is that for a wide range of thresholds the use of the DMSS model reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients. This point does not rely on the results based on the values of the thresholds that maximize average CERs.

0.03%, and 0.06% with 60 draws; and (b) 0.32%, 0.03%, and 0.76% with 120 draws.

We now examine the size of the risk aversion coefficients implied by optimal portfolios within accounts. Panels A, B, and C of Fig. 4 provide box plots of such coefficients for accounts 1, 2, and 3, respectively.[26] Columns (1) and (2) use the thresholds in the first three rows of Table 2A and, respectively, 60 and 120 draws. In each panel, the median coefficients for accounts 1, 2, and 3 notably exceed the risk aversion coefficients that are used to compute their average CERs (i.e., 4, 3, and 1, respectively).[27] Also, note the wide range of implied risk aversion coefficients in each column. Hence, when estimation risk is present, the use of the DMSS model with fixed thresholds differs considerably from the use of the MV model in which the risk aversion coefficient is fixed. Similar results hold in columns (3) and (4) as well as columns (5) and (6), which use the same thresholds as the middle and last three rows of Table 2A, respectively.

4.2.2 Variable thresholds

Of particular interest is the out-of-sample performance of optimal portfolios in the MV model. As noted earlier, optimal portfolios within accounts with variable thresholds coincide with optimal portfolios in the MV model. Hence, we now assess the out-of-sample performance of the former portfolios.

In the first three rows of Table 2C, variable thresholds are set so that the implied risk aversion coefficients of the optimal portfolios within accounts 1, 2, and 3 are exogenously given by, respectively, 4, 3, and 1. As with fixed thresholds, we use risk aversion coefficients of 4, 3, and 1 to compute the average CERs for accounts 1, 2, and 3, respectively. Note that the resulting average CERs are smaller than those of optimal portfolios within accounts for the fixed thresholds in Table

[26] These and subsequent box plots exclude outliers (if any) via Winsorization. Here, an outlier is defined as a value that is above (below) the upper (lower) quartile by an amount that exceeds 1.5 times the size of the interquartile range. Note that the three horizontal lines in a box represent the lower quartile, median, and upper quartile. The dashed vertical lines extending from each end of the box show the range. Hence, the horizontal line at the bottom (top) of the lower (upper) dashed vertical line represents the lowest (highest) value.

[27] Note that the median coefficient for account 1 exceeds that for account 2, which in turn exceeds that for account 3. This result can be understood with three observations. First, the threshold probability of account 1 is lower than that of account 2, which in turn is lower than that of account 3. Second, the threshold return of account 1 is larger than that of account 2, which in turn is larger than that of account 3. Third, as discussed earlier, the probability constraint given by Eq. (3) is tighter when either the threshold probability is lower or the threshold return is larger.

2A. More generally, the former average CERs are smaller than the latter for a wide range of fixed thresholds (see panels A–D of Fig. 3). Consider the case of 60 draws. The first three rows of Table 2C report *negative* average CERs for optimal portfolios in the MV model with risk aversion coefficients of 4, 3, and 1. In comparison, panels A and C of Fig. 3 show *positive* average CERs for optimal portfolios within accounts with a wide range of fixed thresholds. For example, consider account 1. The solid line of panel A shows that the average CER is positive if the threshold probability is $\alpha_1 = 1\%$ and the threshold return H_1 ranges from about -21% to -5%. Also, the solid line of panel C shows that it is positive if the threshold return is $H_1 = -5\%$ and the threshold probability α_1 ranges from 1% to about 14%. Similar results hold for accounts 2 and 3.

Consider now the case of 120 draws. While the first three rows of Table 2C now report positive average CERs for optimal portfolios in the MV model with risk aversion coefficients of 4, 3, and 1, it can be seen that they are still smaller than those of optimal portfolios within accounts for a wide range of fixed thresholds (see panels B and D of Fig. 3). For example, consider the case of account 1. The first row of Table 2C reports an average CER of 1.03% for the optimal portfolio in the MV model for a risk aversion coefficient of 4. In comparison, the solid line of Fig. 3B indicates that the average CER of the optimal portfolio within account 1 exceeds 1.03% if the threshold probability is $\alpha_1 = 1\%$ and the threshold return H_1 ranges from about -21% to -5%. Also, the solid line of Fig. 3D indicates that the average CER of this portfolio exceeds 1.03% if the threshold return is $H_1 = -5\%$ and the threshold probability α_1 ranges from 1% to about 16%. Hence, for a wide range of fixed thresholds, the use of the DMSS model reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients.

The intuition for the reduction in estimation risk is as follows. Since the estimated optimization inputs are imprecise, there is considerable estimation risk when using the MV model with plausible risk aversion coefficients. Hence, the use of larger risk aversion coefficients reduces this risk.[28] As

[28] Recall that the optimal portfolio in the MV model is a combination of $\boldsymbol{w}_0^\varepsilon$ and $\boldsymbol{w}_1^\varepsilon$; see Eq. (7). Noting that $\boldsymbol{w}_0^\varepsilon$ depends solely on $\boldsymbol{\Sigma}^\varepsilon$ and $\boldsymbol{w}_1^\varepsilon$ depends on both $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$, $\boldsymbol{w}_0^\varepsilon$ is subject to less estimation risk than $\boldsymbol{w}_1^\varepsilon$. Since the use of larger risk aversion coefficients leads the optimal portfolio to be closer to $\boldsymbol{w}_0^\varepsilon$ and $\boldsymbol{w}_0^\varepsilon$ is subject to less estimation risk than $\boldsymbol{w}_1^\varepsilon$, the use of

noted earlier, the use of the DMSS model with the fixed thresholds in Table 2A is equivalent to the use of the MV model with larger risk aversion coefficients. Therefore, the use of the former model with these thresholds reduces estimation risk relative to the use of the latter with plausible risk aversion coefficients.

We emphasize that the increase in average CER arising from using the DMSS model instead of the MV model depends on thresholds, risk aversion coefficients, and the number of draws.[29] For exogenous thresholds as well as risk aversion coefficients of 4, 3, and 1, the increases are, respectively: (a) 1.70%, 2.41%, and 7.08% with 60 draws; and (b) 0.19%, 0.66%, and 1.14% with 120 draws.[30] Also, for exogenous threshold probabilities and endogenous threshold returns as well as risk aversion coefficients of 4, 3, and 1, the increases are, respectively: (a) 1.83%, 2.43%, and 7.23% with 60 draws; and (b) 0.52%, 0.69%, and 2.04% with 120 draws. Similar results hold with endogenous threshold probabilities and exogenous threshold returns.

Fig. 5 shows the relation between the average CERs of optimal portfolios within accounts and the implied risk aversion coefficients associated with the variable thresholds. Panels A and B, C and D, and E and F consider accounts 1, 2, and 3, respectively. The number of draws is: (a) 60 in panels A, C, and E; and (b) 120 in panels B, D, and F. With the exception of the case involving account 3 and 120 draws (see panel F), average CERs are negative if implied risk aversion coefficients are sufficiently small. As these coefficients increase, average CERs at first increase sharply for all accounts, but then either remain at a roughly constant positive level for accounts 1 and 2 (see panels A–D) or decrease to a lower positive level for account 3 (see panels E and F).

In the last three rows of Table 2C, thresholds are endogenously set by maximizing the average CERs of optimal portfolios within accounts. As before, we use risk aversion coefficients of 4, 3,

such coefficients reduces estimation risk.

[29] Note that there could be a *reduction* in average CER. For example, in the case of 120 draws, the average CER of the optimal portfolio within account 2 is *negative* if the threshold probability and return are, respectively, 5% and −25% (see the dashed line of Fig. 3B), whereas that of the optimal portfolio in the MV model is *positive* if the risk aversion coefficient is 3 (see the second row of Table 2C).

[30] For example, using a risk aversion coefficient of 4 and 60 draws, the increase is 1.14% − (−0.56%) = 1.70% where 1.14% and −0.56% are from the first row of, respectively, Tables 2A and 2C.

and 1 to compute the average CERs for accounts 1, 2, and 3, respectively. Observe that these coefficients generally differ from the implied risk aversion coefficients of optimal portfolios within accounts with endogenous variable thresholds. In setting such thresholds, we compute the average CER for each element in an appropriate grid of implied risk aversion coefficients and then identify the element that leads to the largest average CER as well as the associated thresholds. With these thresholds, the implied risk aversion coefficients of the optimal portfolios within accounts 1, 2, and 3 exceed, respectively, 4, 3, and 1. By design, the resulting average CERs exceed those of optimal portfolios in the MV model with risk aversion coefficients of 4, 3, and 1 (in Table 2C, compare the last and first three rows). More generally, for variable thresholds associated with a wide range of implied risk aversion coefficients, the average CERs of optimal portfolios within accounts (see panels A–F of Fig. 5) exceed those of optimal portfolios in the MV model with risk aversion coefficients of 4, 3, and 1. In panels A–E, if the implied risk aversion coefficient is larger than the risk aversion coefficient, then the average CER of the optimal portfolio within a given account exceeds that of the optimal portfolio in the MV model. In panel F, if the implied risk aversion coefficient is strictly between 1 and 6, then the average CER of the optimal portfolio within account 3 exceeds that of the optimal portfolio in the MV model with a risk aversion coefficient of 1. Panels A–F also show that the increase in average CER arising from using the DMSS model with variable thresholds associated with a given implied risk aversion coefficient instead of the MV model with a plausible risk aversion coefficient depends on the size of these two coefficients as well as the number of draws.

In assessing the statistical significance of the difference between the distributions of CERs for optimal portfolios within accounts and optimal portfolios in the MV model with plausible risk aversion coefficients, we utilize: (i) the two-sample Kolmogorov-Smirnov test and (ii) the Wilcoxon rank sum test. For example, consider the case of 60 draws, fixed exogenous thresholds, and account 1. Using (i), we test the null hypothesis that the cdf of CERs for the optimal portfolio within account 1 with a threshold probability of 1% and a threshold return of −5% coincides with the cdf of CERs for the optimal portfolio in the MV model with a risk aversion coefficient of 4. While the former CERs are used in the first row of Table 2A, the latter are used in the first row of Table 2C (see the first set of either two or three columns to the right of the 'Account' column). The alternative hypothesis is that the two cdfs differ. Similarly, using (ii), we test the null hypothesis that the median of the distribution of CERs for the optimal portfolio within account 1 with a threshold probability of 1% and a threshold return of −5% coincides with the median of CERs for the optimal portfolio in the MV model with a risk aversion coefficient of 4. The alternative hypothesis is that the two medians differ. We also conduct tests for the cases of either: (a) 120 draws; (b) fixed exogenous threshold probabilities and endogenous threshold returns, fixed exogenous threshold returns and endogenous threshold probabilities, or variable endogenous thresholds; and (c) accounts 2 or 3. Since we use two test statistics, two numbers of draws, four types of thresholds, and three accounts, we conduct 48 [= 2 × 2 × 4 × 3] tests. For all of these tests, we find that the null hypothesis is rejected at the 1% level. Hence, there is statistical significance to the result that the use of the DMSS model reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients.[31]

4.3. Aggregate portfolio

Next, we assess the out-of-sample performance of aggregate portfolios with the fixed thresholds from Table 2A. Table 3A presents their average CERs using a risk aversion coefficient of 2.[32] Note that the average CERs in the second row (with exogenous threshold probabilities and endogenous threshold returns) exceed those in the third row (with endogenous threshold probabilities and ex-

[31] Additionally, we find that the weights of optimal portfolios within accounts (with the thresholds used in Table 2A as well as the last three rows of Table 2C) are more stable than those of optimal portfolios in the MV model (with the risk aversion coefficients used in the first three rows of Table 2C). In assessing the stability of the weights of the portfolio within any given account $m \in \{1, 2, 3\}$ with simulated data, we compute $\frac{\sum_{s=1}^{1000} \left| \boldsymbol{w}^{\varepsilon}_{m,s+1} - \boldsymbol{w}^{\varepsilon}_{m,s} \right| / \sqrt{8}}{999}$ where $|\cdot|$ denotes the Euclidean norm (we proceed similarly when either assessing optimal portfolios in the MV model or using empirical data). This finding suggests that the transaction costs arising from implementing the DMSS model are smaller than those arising from implementing the MV model. A detailed comparison of the transaction costs arising from implementing these two models is left for future research.

[32] While we obtain similar results using other reasonable values for this coefficient, a value of 2 can be justified as follows. In the absence of estimation risk, the risk aversion coefficient implied by the aggregate portfolio is $\gamma_a = 1/(\sum_{m=1}^{M} y_m/\gamma_m)$. Recalling that (1) there are $M = 3$ accounts, (2) the fractions of wealth in the accounts are given by $(y_1, y_2, y_3) = (0.4, 0.3, 0.3)$, and (3) average CERs of optimal portfolios within accounts are determined by using risk aversion coefficients of $(\gamma_1, \gamma_2, \gamma_3) = (4, 3, 1)$, we have $1/(\sum_{m=1}^{M} y_m/\gamma_m) = 1/(0.4/4 + 0.3/3 + 0.3/1) = 2$.

ogenous threshold returns), which in turn exceed those in the first row (with exogenous thresholds). As with optimal portfolios within accounts, average CERs increase in the number of draws.

Fig. 6 provides box plots of the risk aversion coefficients implied by aggregate portfolios for the fixed thresholds in Table 3A. Note that the median coefficients with exogenous thresholds (in columns (1)–(2)) exceed those with exogenous threshold probabilities and endogenous threshold returns (in columns (3)–(4)) as well as those with endogenous threshold probabilities and exogenous threshold returns (in columns (5)–(6)). In each column, the median coefficient exceeds the risk aversion coefficient of 2 that is used to compute the average CERs of aggregate portfolios.

Table 3C assesses the out-of-sample performance of aggregate portfolios with the variable thresholds from Table 2C. Note that the average CERs in the first row (with exogenous implied risk aversion coefficients) are smaller than those in the second row (with endogenous coefficients). As with fixed thresholds, average CERs increase in the number of draws.

5. Empirical data

We now use empirical data. As we explain shortly, the use of empirical data allows us to consider the case where the first two moments of the distribution of asset returns possibly vary over time.

5.1. Methodology

Our methodology takes seven steps. Steps 1 and 2 are identical to those when using simulated data (see Section 4.1). In step 3, we use the 60 vectors of monthly asset returns for the time period 1978–1982 (one vector for each month) to determine the estimated optimization inputs. In step 4, we use these inputs to find the composition of optimal portfolios within accounts and the aggregate portfolio, which are assumed to be held in 1983.[33] In step 5, we repeat steps 3 and 4 (31 times) by using monthly asset returns in the time periods 1979–1983, ..., 2009–2013 to determine

[33] For brevity, we omit the figures based on empirical data that examine: (1) the existence of optimal portfolios within accounts with fixed thresholds; (2) average CERs of optimal portfolios within accounts with either fixed or variable thresholds; (3) implied risk aversion coefficients of optimal portfolios within accounts with fixed thresholds; and (4) implied risk aversion coefficients of aggregate portfolios. The results in these figures are similar to those reported for the corresponding Figs. 2–6 based on simulated data.

the estimated optimization inputs. In step 6, for each account we compute the *average CER* of the 32 optimal portfolios within the account (one portfolio for each time period). Let $\boldsymbol{w}^{\varepsilon}_{m,t}$ denote the optimal portfolio within account m held in year t for $m = 1, 2, 3$ and $t = 1983, ..., 2014$. For any account $m \in \{1, 2, 3\}$ and any risk aversion coefficient $\gamma > 0$, the average CER of portfolios $\{\boldsymbol{w}^{\varepsilon}_{m,t}\}_{t=1983}^{2014}$ is $\overline{CER}^{\varepsilon}_{m,\gamma} \equiv \frac{\sum_{t=1983}^{2014} E[r_{w^{\varepsilon}_{m,t}}] - \frac{\gamma}{2}(\sigma[r_{w^{\varepsilon}_{m,t}}])^2}{32}$. Here, $E[r_{\boldsymbol{w}^{\varepsilon}_{m,t}}]$ and $\sigma[r_{\boldsymbol{w}^{\varepsilon}_{m,t}}]$ are obtained by using the monthly returns of portfolio $\boldsymbol{w}^{\varepsilon}_{m,t}$ in year t. Hence, the first two moments of the distribution of asset returns possibly vary over time. Similarly, we compute the average CER of the 32 aggregate portfolios (again, one portfolio for each time period). In step 7, we repeat steps 3–6 by using 120 (instead of 60) months to determine the estimated optimization inputs.

5.2. Optimal portfolios within accounts

Tables 4A and 4C report the average CERs of optimal portfolios within accounts with, respectively, fixed and variable thresholds. Compared to Tables 2A and 2C where simulated data are used, the respective average CERs are smaller. However, the extent to which average CERs associated with the use of the DMSS model (see Table 4A) exceed those associated with the use of the MV model with plausible risk aversion coefficients (see the first three rows of Table 4C) is larger.[34]

The quantitative differences between the results with empirical and simulated data can be understood as follows. First, the values of the estimated optimization inputs when using empirical data differ from those when using simulated data. Second, while the first two moments of the distribution of asset returns are assumed to vary over time when computing average CERs with empirical data, they are assumed to be constant with simulated data.

[34] When empirical data are used, we do not conduct the two-sample Kolmogorov-Smirnov and Wilcoxon rank sum tests to assess the statistical significance of the difference between the distributions of CERs for optimal portfolios within accounts and optimal portfolios in the MV model. The reason why we do not conduct these tests is that the assumption of random sample selection (see, e.g., Sheskin (2011)) does not exactly hold. For example, consider the case of 60 months being used to determine the estimated optimization inputs and account 1. The optimal portfolios for this account that are held in 1983 and 1984 are obtained by using estimated optimization inputs based on asset returns for the time periods of, respectively, 1978–1982 and 1979–1983 (see Section 5.1). Since the time periods overlap in four (out of five) years, the portfolio held in 1983 is related to that held in 1984. Hence, the CER of the former portfolio is also related to the CER of the latter.

5.3. Aggregate portfolio

Table 5A and 5C report the average CERs of aggregate portfolios with, respectively, fixed and variable thresholds. Compared to Table 3A and 3C where simulated data are used, the respective average CERs are smaller. Focusing on variable thresholds, note that the difference between the average CERs in the two rows of Table 5C is larger than that in the two rows of Table 3C.

6. Disallowing short selling

We now extend Sections 4 and 5 to the case where short selling is disallowed.

6.1. Simulated data

Next, we use our simulated data.

6.1.1. Optimal portfolios within accounts

Consider fixed thresholds. Compared to panels A and B of Fig. 2 where short selling is allowed, panels C and D indicate that there is a larger set of thresholds for which optimal portfolios within accounts exist in all simulations. When allowed, they exist if and only if (1) threshold probabilities are sufficiently low *and* (2) threshold returns are sufficiently small. However, when disallowed, they exist if and only if (2) holds.

Table 2B assesses the out-of-sample performance of optimal portfolios within accounts. Compared to Table 2A where short selling is allowed, there are two main differences. First, average CERs are smaller. Second, there are smaller increases (if any) in average CERs arising from using either endogenous threshold probabilities and exogenous threshold returns (in the middle three rows of Table 2B) or endogenous threshold probabilities and exogenous threshold returns (in the last three rows) instead of exogenous thresholds (in the first three rows). These differences can be seen by examining Fig. 3. Specifically, average CERs with short selling disallowed (in panels E–H) are smaller than those with short selling allowed and endogenous thresholds (in panels A–D, see the average CERs at the points where the lines peak). Also, unlike in the latter panels, the relation between average CERs and thresholds is almost flat in the former.

Panels D–F of Fig. 4 provide box plots of risk aversion coefficients implied by optimal portfolios within accounts.[35] In nearly all cases, median coefficients are smaller than those in panels A–C where short selling is allowed (the median coefficients in columns (4) and (6) of panel F are slightly larger than those in, respectively, columns (4) and (6) of panel C).[36]

Consider now variable thresholds. Table 2D assesses the out-of-sample performance of optimal portfolios within accounts. Compared to the case where short selling is allowed, three differences are worth noting. First, average CERs are smaller with few exceptions. Specifically, when the number of draws is 60, the average CERs in the first three rows of Table 2D exceed those in the first three rows of Table 2C. Second, increases in average CERs (if any) arising from using endogenous thresholds are also smaller (compare the difference between the last and first three rows of Table 2D with the difference in Table 2C).[37] Third, while there is still an increase in average CER arising from using the DMSS model with fixed thresholds instead of the MV model with a plausible risk aversion coefficient in all but one case, it is considerably smaller than that when short selling is allowed. This case involves using 120 draws, account 1, and exogenous thresholds. In such a case, the average CER when using the DMSS model with thresholds of $\alpha_1 = 1\%$ and $H_1 = -5\%$ equals that when using the MV model with a risk aversion coefficient of 4 (compare the first line of Tables 2B and 2D). In other cases, the fact that there are smaller increases in average CERs (relative to when short selling is allowed) can be seen by noting that the differences between the average CERs in Table 2B and those in the first three rows of Table 2D are smaller than the differences between the average CERs in Table 2A and those in the first three rows of Table 2C.

[35] When short selling is disallowed, the optimal portfolio within a given account has maximum estimated expected return among all feasible portfolios in the cases where either the threshold probability is sufficiently high or the threshold return is sufficiently small. In such cases, a hypothetical MV investor would select this portfolio if he or she has a risk aversion coefficient between zero and some positive value. In reporting its implied risk aversion coefficient, we follow the convention of using the largest risk aversion coefficient for which the investor would select it.

[36] Note that the minimum coefficient is essentially zero. Such a coefficient can be obtained if the optimal portfolio within the corresponding account is the portfolio with maximum estimated expected return.

[37] The fact that increases in average CERs are smaller can be seen by inspecting Fig. 5. In panels A–F, average CERs are very sensitive to implied risk aversion coefficients if such coefficients are relatively small. In panels G–L, average CERs are not very sensitive to implied risk aversion coefficients regardless of the size of such coefficients. Unlike when short selling is allowed, variable thresholds can be set so that the risk aversion coefficient implied by the optimal portfolio within a given account is zero. Hence, the x-axis of panels G–L of Fig. 5 ranges from zero to 20 (instead of ranging from one to 20 as in panels A–F).

The result that there are smaller increases in average CERs arising from using the DMSS model with fixed thresholds instead of the MV model with plausible risk aversion coefficients relative to the case when short selling is allowed can be understood as follows. As noted earlier, disallowing short selling reduces estimation risk in the MV model; see, e.g., Jagganathan and Ma (2002). Hence, when short selling is disallowed, the average CERs associated with the use of the MV model with plausible risk aversion coefficients are closer to those associated with the use of the DMSS model with fixed thresholds relative to the case when it is allowed.

As before, we utilize the two-sample Kolmogorov-Smirnov and Wilcoxon rank sum tests to assess the statistical significance of the difference between the distributions of CERs for optimal portfolios within accounts and optimal portfolios in the MV model with plausible risk aversion coefficients. In 36 of the 48 tests, we find that the null hypothesis is rejected at the 1% level. Hence, there is statistical significance to the result that the use of the DMSS model reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients. However, the statistical significance is weaker than in the preceding case where short selling is allowed (in which the null hypothesis is rejected at the 1% level in all 48 tests).

6.1.2. Aggregate portfolio

Table 3B reports the average CERs of aggregate portfolios with fixed thresholds.[38] The results differ from those in Table 3A where short selling is allowed in two respects. First, average CERs are smaller. Second, in the case of 120 draws, there are smaller increases in average CERs arising from using either endogenous threshold probabilities and exogenous threshold returns (in the second row of Table 3B) or endogenous threshold probabilities and exogenous threshold returns (in the third row) instead of exogenous thresholds (in the first row).

Table 3D reports the average CERs of aggregate portfolios with variable thresholds. As with

[38] Note that the aggregate portfolio might lie away from the estimated MV frontier when short selling is disallowed (DMSS make a similar point when estimation risk is absent). By construction, an MV investor would never select a portfolio that lies away from the estimated MV frontier. Hence, the implied risk aversion coefficient of the aggregate portfolio cannot be determined if it lies away from the frontier. Accordingly, we do not report its implied risk aversion coefficient when short selling is disallowed.

fixed thresholds, the results differ from those in Table 3C where short selling is allowed in two respects. First, average CERs are smaller when using endogenous implied risk aversion coefficients; compare the second row of Tables 3D and 3C. Second, there are smaller increases in average CERs (if any) arising from using endogenous implied risk aversion coefficients; note that the differences in average CERs (if any) in the two lines of Table 3D are smaller than the differences in average CERs in the two lines of Table 3C.

6.2. Empirical data

We now use empirical data.

6.2.1. Optimal portfolios within accounts

Tables 4B and 4D show the average CERs of optimal portfolios within accounts with, respectively, fixed and variable thresholds. Compared to the corresponding Tables 2B and 2D where simulated data are used, it can be seen that average CERs are smaller.

6.2.2. Aggregate portfolio

Tables 5B and 5D present the average CERs of aggregate portfolios with, respectively, fixed and variable thresholds. Compared to the corresponding Tables 3B and 3D where simulated data are used, note that average CERs are smaller.

7. Practical implications

This section summarizes two practical implications of our results for asset managers.[39] First, thresholds need to be carefully set so that optimal portfolios within accounts (and thus the aggregate portfolio) exist. When short selling is allowed, they exist if and only if (1) threshold probabilities are sufficiently low *and* (2) threshold returns are sufficiently small. When short selling is disallowed, they exist if and only if (2) holds.

[39] Here, we use the terms 'asset managers' in a broad sense. Hence, examples of asset managers here include: (1) portfolio managers in mutual and hedge funds; (2) firms or individuals who manage portfolios on behalf of institutional investors such as endowments and pension plans; (3) financial advisers who manage portfolios on behalf of retail investors; and (4) investors who manage their own portfolios.

Second, if short selling is allowed, then there is a wide range of thresholds for which the use of the DMSS model notably reduces estimation risk relative to the use of the MV model with a plausible risk aversion coefficient. While typically there is still a reduction in estimation risk if short selling is disallowed, this reduction is considerably smaller.

8. Conclusion

Das, Markowitz, Scheid, and Statman (2010, DMSS) develop a behavioral-based portfolio selection model in which the investor divides his or her wealth among accounts with motives such as retirement and bequest. For each account, short selling is allowed and the optimal portfolio has maximum expected return subject to: (1) fully investing the wealth in the account; and (2) the probability of the account's return being less than or equal to some threshold return not exceeding some threshold probability. Reflecting different account motives, thresholds possibly vary across accounts. Nevertheless, optimal portfolios within accounts and the corresponding aggregate portfolio are on the MV frontier.

Our paper complements DMSS by recognizing estimation risk. We begin by theoretically characterizing the existence and composition of optimal portfolios within accounts and the aggregate portfolio. Their existence is found to depend on the thresholds and estimated optimization inputs. When such portfolios exist, they are on the estimated MV frontier.

Using simulated and empirical data, we then assess the out-of-sample performance of optimal portfolios within accounts. We find that there is a wide range of thresholds for which optimal portfolios within accounts notably outperform optimal portfolios in the MV model with plausible risk aversion coefficients. However, when short selling is disallowed, the former portfolios typically still outperform the latter but to a considerably lesser extent. Hence, the use of the DMSS model reduces estimation risk relative to the use of the MV model with plausible risk aversion coefficients.

Since DMSS argue in favor of using their model, in the face of estimation risk an assessment of its out-of-sample performance is of practical interest. While our analysis suggests that the DMSS model is a valid approach to cope with such risk, we do not claim that it is the best approach. For example, the literature suggests the use of either the estimated minimum-variance portfolio or the equally-weighted portfolio. Using simulated and empirical data, we report that these portfolios are outperformed by optimal portfolios within accounts with some exceptions. However, a detailed analysis of the relative performance of such portfolios is left for future research.

In assessing the relative performance of the DMSS and MV models, we use classical estimators for the optimization inputs. An assessment of their relative performance with other estimators (e.g., Bayesian or robust) is also left for further research.

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Table 1: Summary statistics for asset returns

Using monthly data during 1978–2014, this table reports average returns, standard deviations, VaRs, and correlation coefficients for: (a) Treasury bonds; (b) corporate bonds; and (c) the six size/book equity-to-market Fama-French equity portfolios. Returns on the Treasury and corporate bonds are obtained from Bloomberg by using the Bank of America Merrill Lynch Treasury and corporate bond indices, respectively. Returns on the Fama-French equity portfolios are obtained from Kenneth French's website. These portfolios result from sorting stocks along the dimensions of: (i) market capitalization (small and big); and (ii) book equity-to-market equity ratio (low, intermediate, and high). Average returns, standard deviations, and VaRs are reported in percentage points.

	Treasury	Corporate	Fama-French equity portfolios					
	bonds	bonds	Small/low	Small/int.	Small/high	Big/low	Big/int.	Big/high
Average return	0.63	0.68	0.97	1.39	1.47	1.05	1.07	1.12
Standard deviation	1.61	1.99	6.81	5.24	5.33	4.68	4.42	4.60
VaR: 90% confidence level	1.43	1.87	7.75	5.33	5.36	4.95	4.59	4.78
95% confidence level	2.01	2.60	10.23	7.23	7.30	6.65	6.20	6.45
99% confidence level	3.11	3.96	14.87	10.81	10.93	9.84	9.21	9.59
	Correlation coefficient							
Treasury bonds	1.00	0.85	−0.03	0.00	0.01	0.10	0.09	0.07
Corporate bonds		1.00	0.18	0.24	0.25	0.30	0.32	0.30
Fama-French equity portfolios: Small/low			1.00	0.93	0.87	0.82	0.73	0.67
Small/int.				1.00	0.97	0.80	0.82	0.80
Small/high					1.00	0.75	0.81	0.83
Big/low						1.00	0.87	0.78
Big/int.							1.00	0.91
Big/high								1.00

Table 2: Average CERs of optimal portfolios within accounts using simulated data

This table shows average CERs of optimal portfolios within accounts using simulated data. The number of draws used to find the estimated optimization inputs is either 60 or 120. While panels A and B use fixed threshold probabilities and returns, panels C and D use variable thresholds. Short selling is allowed (disallowed) in panels A and C (B and D). In the first three rows of panels A and B, threshold probabilities and returns are exogenous. In the next three rows, threshold probabilities are exogenous, whereas threshold returns are endogenously set by maximizing average CERs. Similarly, in the last three rows, threshold returns are exogenous, whereas threshold probabilities are endogenously set by maximizing average CERs. In the first three rows of panels C and D, threshold probabilities and returns are set so that the risk aversion coefficients implied by the optimal portfolios within accounts 1, 2, and 3 are exogenously given by, respectively, 4, 3, and 1. In the last three rows, they are endogenously set by maximizing average CERs. In determining the CERs for accounts 1, 2, and 3, all panels use risk aversion coefficients of, respectively, 4, 3, and 1 (except for the first three rows of panels C and D, these coefficients generally differ from the implied risk aversion coefficients).

	Threshold		Avg.	Threshold		Avg.
	probability (%)	return (%)	CER (%)	probability (%)	return (%)	CER (%)
Account	Number of draws = 60			Number of draws = 120		
Panel A: Fixed thresholds, short selling allowed						
1	1.00	−5.00	1.14	1.00	−5.00	1.22
2	5.00	−8.00	1.46	5.00	−8.00	1.83
3	10.00	−10.00	2.95	10.00	−10.00	3.37
1	1.00	−8.62	1.27	1.00	−11.68	1.55
2	5.00	−6.79	1.48	5.00	−9.70	1.86
3	10.00	−13.40	3.10	10.00	−20.57	4.27
1	4.93	−5.00	1.26	9.36	−5.00	1.54
2	3.34	−8.00	1.49	7.26	−8.00	1.86
3	11.65	−10.00	3.01	18.06	−10.00	4.13
Panel B: Fixed thresholds, short selling disallowed						
1	1.00	−5.00	0.80	1.00	−5.00	0.83
2	5.00	−8.00	0.89	5.00	−8.00	0.97
3	10.00	−10.00	1.13	10.00	−10.00	1.23
1	1.00	−6.82	0.82	1.00	−7.29	0.86
2	5.00	−6.27	0.90	5.00	−7.61	0.97
3	10.00	−5.83	1.13	10.00	−6.25	1.23
1	3.39	−5.00	0.82	4.44	−5.00	0.86
2	2.22	−8.00	0.90	4.26	−8.00	0.97
3	2.12	−10.00	1.14	2.58	−10.00	1.23

	Implied risk aversion coefficient	Avg. CER (%)	Implied risk aversion coefficient	Avg. CER (%)
Account	Number of draws = 60		Number of draws = 120	
Panel C: Variable thresholds, short selling allowed				
1	4.00	−0.56	4.00	1.03
2	3.00	−0.95	3.00	1.17
3	1.00	−4.13	1.00	2.23
1	10.95	1.24	6.98	1.52
2	8.21	1.45	5.24	1.82
3	2.74	3.07	1.75	4.16
Panel D: Variable thresholds, short selling disallowed				
1	4.00	0.79	4.00	0.83
2	3.00	0.88	3.00	0.93
3	1.00	1.12	1.00	1.21
1	4.95	0.79	1.29	0.84
2	1.95	0.89	0.33	0.97
3	0.10	1.13	0.06	1.23

Table 3: Average CERs of aggregate portfolios using simulated data

This table reports average CERs of aggregate portfolios using simulated data. The number of draws used to find the estimated optimization inputs is either 60 or 120. The fractions of wealth in accounts 1, 2, and 3 are, respectively, 40%, 30%, and 30%. While short selling is allowed in panels A and C, it is disallowed in panels B and D. Panels A, B, C, and D use the same threshold probabilities and returns as, respectively, panels A, B, C, and D of Table 2.

Threshold						Avg. CER (%)	Threshold						Avg. CER (%)
probability (%)			return (%)				probability (%)			return (%)			
Account							Account						
1	2	3	1	2	3		1	2	3	1	2	3	
Number of draws = 60							Number of draws = 120						
Panel A: Fixed thresholds, short selling allowed													
1.00	5.00	10.00	−5.00	−8.00	−10.00	1.85	1.00	5.00	10.00	−5.00	−8.00	−10.00	2.10
1.00	5.00	10.00	−8.62	−6.79	−13.40	1.90	1.00	5.00	10.00	−11.68	−9.70	−20.57	2.47
4.93	3.34	11.65	−5.00	−8.00	−10.00	1.87	9.36	7.26	18.06	−5.00	−8.00	−10.00	2.43
Panel B: Fixed thresholds, short selling disallowed													
1.00	5.00	10.00	−5.00	−8.00	−10.00	0.97	1.00	5.00	10.00	−5.00	−8.00	−10.00	1.04
1.00	5.00	10.00	−6.82	−6.27	−5.83	0.99	1.00	5.00	10.00	−7.29	−7.61	−6.25	1.07
3.39	2.22	2.12	−5.00	−8.00	−10.00	0.99	4.44	4.26	2.58	−5.00	−8.00	−10.00	1.07

Implied risk aversion coefficient			Avg. CER (%)	Implied risk aversion coefficient			Avg. CER (%)
Account				Account			
1	2	3		1	2	3	
Number of draws = 60				Number of draws = 120			
Panel C: Variable thresholds, short selling allowed							
4.00	3.00	1.00	−1.74	4.00	3.00	1.00	1.44
10.95	8.21	2.74	1.86	6.98	5.24	1.75	2.41
Panel D: Variable thresholds, short selling disallowed							
4.00	3.00	1.00	0.98	4.00	3.00	1.00	1.04
4.95	1.95	0.10	0.98	1.29	0.33	0.06	1.09

Table 4: Average CERs of optimal portfolios within accounts using empirical data

This table shows average CERs of optimal portfolios within accounts using empirical data. The number of months in the periods used to find the estimated optimization inputs is either 60 or 120. While panels A and B use fixed threshold probabilities and returns, panels C and D use variable thresholds. Short selling is allowed (disallowed) in panels A and C (B and D). In the first three rows of panels A and B, threshold probabilities and returns are exogenous. In the next three rows, threshold probabilities are exogenous, whereas threshold returns are endogenously set by maximizing average CERs. Similarly, in the last three rows, threshold returns are exogenous, whereas threshold probabilities are endogenously set by maximizing average CERs. In the first three rows of panels C and D, threshold probabilities and returns are set so that the risk aversion coefficients implied by the optimal portfolios within accounts 1, 2, and 3 are exogenously given by, respectively, 4, 3, and 1. In the last three rows, they are endogenously set by maximizing average CERs. In determining the CERs for accounts 1, 2, and 3, all panels use risk aversion coefficients of, respectively, 4, 3, and 1 (except for the first three rows of panels C and D, these coefficients generally differ from the implied risk aversion coefficients).

	Threshold		Avg.	Threshold		Avg.
	probability (%)	return (%)	CER (%)	probability (%)	return (%)	CER (%)
Account	Number of months = 60			Number of months = 120		
Panel A: Fixed thresholds, short selling allowed						
1	1.00	−5.00	1.00	1.00	−5.00	1.10
2	5.00	−8.00	0.47	5.00	−8.00	1.46
3	10.00	−10.00	1.50	10.00	−10.00	3.08
1	1.00	−5.03	1.00	1.00	−8.60	1.23
2	5.00	−3.52	1.13	5.00	−7.00	1.48
3	10.00	−5.99	1.99	10.00	−14.74	3.34
1	1.08	−5.00	1.06	5.73	−5.00	1.35
2	0.42	−8.00	1.20	3.82	−8.00	1.61
3	5.61	−10.00	2.19	15.12	−10.00	3.70
Panel B: Fixed thresholds, short selling disallowed						
1	1.00	−5.00	0.54	1.00	−5.00	0.68
2	5.00	−8.00	0.67	5.00	−8.00	0.79
3	10.00	−10.00	0.92	10.00	−10.00	1.06
1	1.00	−1.88	0.64	1.00	−8.88	0.68
2	5.00	−12.86	0.69	5.00	−8.01	0.79
3	10.00	−10.64	0.92	10.00	−8.51	1.06
1	23.80	−5.00	0.58	9.11	−5.00	0.68
2	15.78	−8.00	0.69	4.76	−8.00	0.79
3	9.85	−10.00	0.92	3.81	−10.00	1.06

	Implied risk aversion coefficient	Avg. CER (%)	Implied risk aversion coefficient	Avg. CER (%)
Account	Number of months = 60		Number of months = 120	
Panel C: Variable thresholds, short selling allowed				
1	4.00	−5.85	4.00	−0.21
2	3.00	−8.08	3.00	−0.46
3	1.00	−25.94	1.00	−2.57
1	21.51	1.00	9.74	1.26
2	16.36	1.12	7.33	1.50
3	5.61	1.99	2.46	3.36
Panel D: Variable thresholds, short selling disallowed				
1	4.00	0.43	4.00	0.63
2	3.00	0.51	3.00	0.74
3	1.00	0.73	1.00	1.04
1	68.69	0.62	3.20	0.67
2	0.00	0.70	0.00	0.80
3	0.00	0.93	0.03	1.06

Table 5: Average CERs of aggregate portfolios using empirical data

This table reports average CERs of aggregate portfolios using empirical data. The number of months in the periods used to find the estimated optimization inputs is either 60 or 120. The fractions of wealth in accounts 1, 2, and 3 are, respectively, 40%, 30%, and 30%. While short selling is allowed in panels A and C, it is disallowed in panels B and D. Panels A, B, C, and D use the same threshold probabilities and returns as, respectively, panels A, B, C, and D of Table 4.

Threshold						Avg.	Threshold						Avg.
probability (%)			return (%)			CER	probability (%)			return (%)			CER
Account						(%)	Account						(%)
1	2	3	1	2	3		1	2	3	1	2	3	
Number of months = 60							Number of months = 120						
Panel A: Fixed thresholds, short selling allowed													
1.00	5.00	10.00	−5.00	−8.00	−10.00	1.11	1.00	5.00	10.00	−5.00	−8.00	−10.00	1.86
1.00	5.00	10.00	−5.03	−3.52	−5.99	1.35	1.00	5.00	10.00	−8.60	−7.00	−14.74	1.95
1.08	0.42	5.61	−5.00	−8.00	−10.00	1.45	5.73	3.82	15.12	−5.00	−8.00	−10.00	2.14
Panel B: Fixed thresholds, short selling disallowed													
1.00	5.00	10.00	−5.00	−8.00	−10.00	0.74	1.00	5.00	10.00	−5.00	−8.00	−10.00	0.87
1.00	5.00	10.00	−1.88	−12.86	−10.64	0.79	1.00	5.00	10.00	−8.88	−8.01	−8.51	0.91
23.80	15.78	9.85	−5.00	−8.00	−10.00	0.81	9.11	4.76	3.81	−5.00	−8.00	−10.00	0.92

Implied risk aversion coefficient			Avg.	Implied risk aversion coefficient			Avg.
Account			CER	Account			CER
1	2	3	(%)	1	2	3	(%)
Number of months = 60				Number of months = 120			
Panel C: Variable thresholds, short selling allowed							
4.00	3.00	1.00	−12.54	4.00	3.00	1.00	−0.99
21.51	16.34	5.61	1.34	9.74	7.33	2.46	1.97
Panel D: Variable thresholds, short selling disallowed							
4.00	3.00	1.00	0.61	4.00	3.00	1.00	0.84
68.69	0.00	0.00	0.79	3.20	0.00	0.03	0.90

Fig. 1: Existence of the optimal portfolio within a given account

The curve shows the portfolios on the estimated MV frontier when short selling is allowed. Fix any account $m \in \{1, ..., M\}$ with threshold probability and return given by, respectively, α_m and H_m. The line has intercept H_m and slope z_{α_m}. Portfolios on or above this line satisfy constraint (6), whereas portfolios below it do not. Note that the constraint is tightened if either α_m decreases or H_m increases. Recall that α^ε is defined in Eq. (9). Also, $H^\varepsilon_{\alpha_m}$ is given by Eq. (10) with $\alpha = \alpha_m$. When $\alpha_m \geq \alpha^\varepsilon$, the optimal portfolio within account m does not exist regardless of the threshold return (see panels A and B). When $\alpha_m < \alpha^\varepsilon$, the optimal portfolio within account m does not exist if $H_m > H^\varepsilon_{\alpha_m}$ (see panel C), but it exists if either $H_m = H^\varepsilon_{\alpha_m}$ (see panel D) or $H_m < H^\varepsilon_{\alpha_m}$ (see panel E). In panels D and E, the optimal portfolio within account m is represented by point p_m. In panel D, this portfolio is located where the line is tangent to the curve. In panel E, the portfolio is located where the line crosses the top half of the curve.

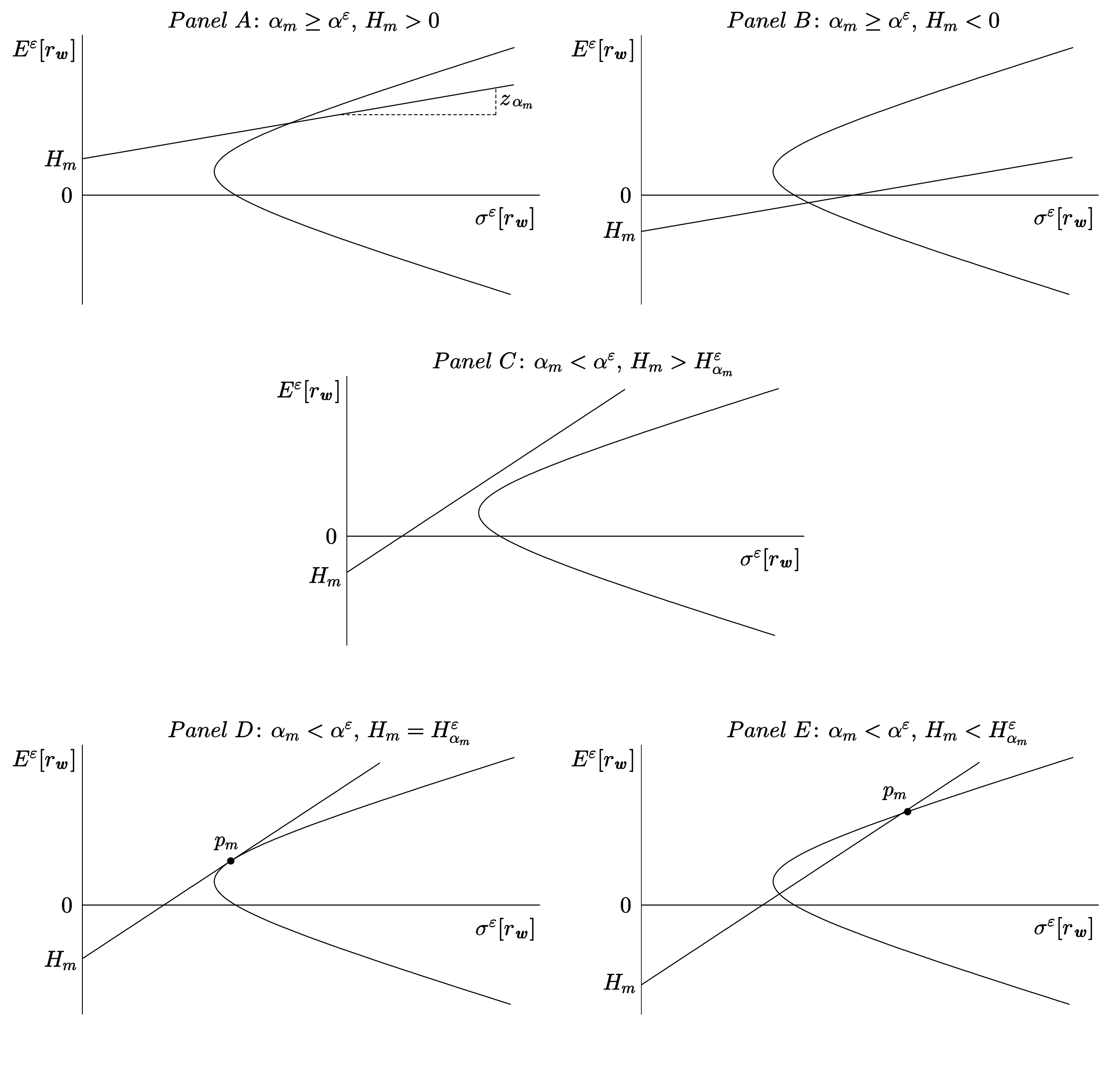


Fig. 2: Existence of optimal portfolios within accounts using fixed threshold probabilities and returns as well as simulated data

This figure examines the existence of optimal portfolios within accounts using fixed threshold probabilities and returns as well as simulated data. The number of draws used to find the estimated optimization inputs, N_{draws}, is either 60 or 120. Each panel reports the fraction of simulations for which the optimal portfolios within any given account m exist for various values of threshold probability α_m and threshold return H_m. While short selling is allowed in panels A and B, it is disallowed in panels C and D.

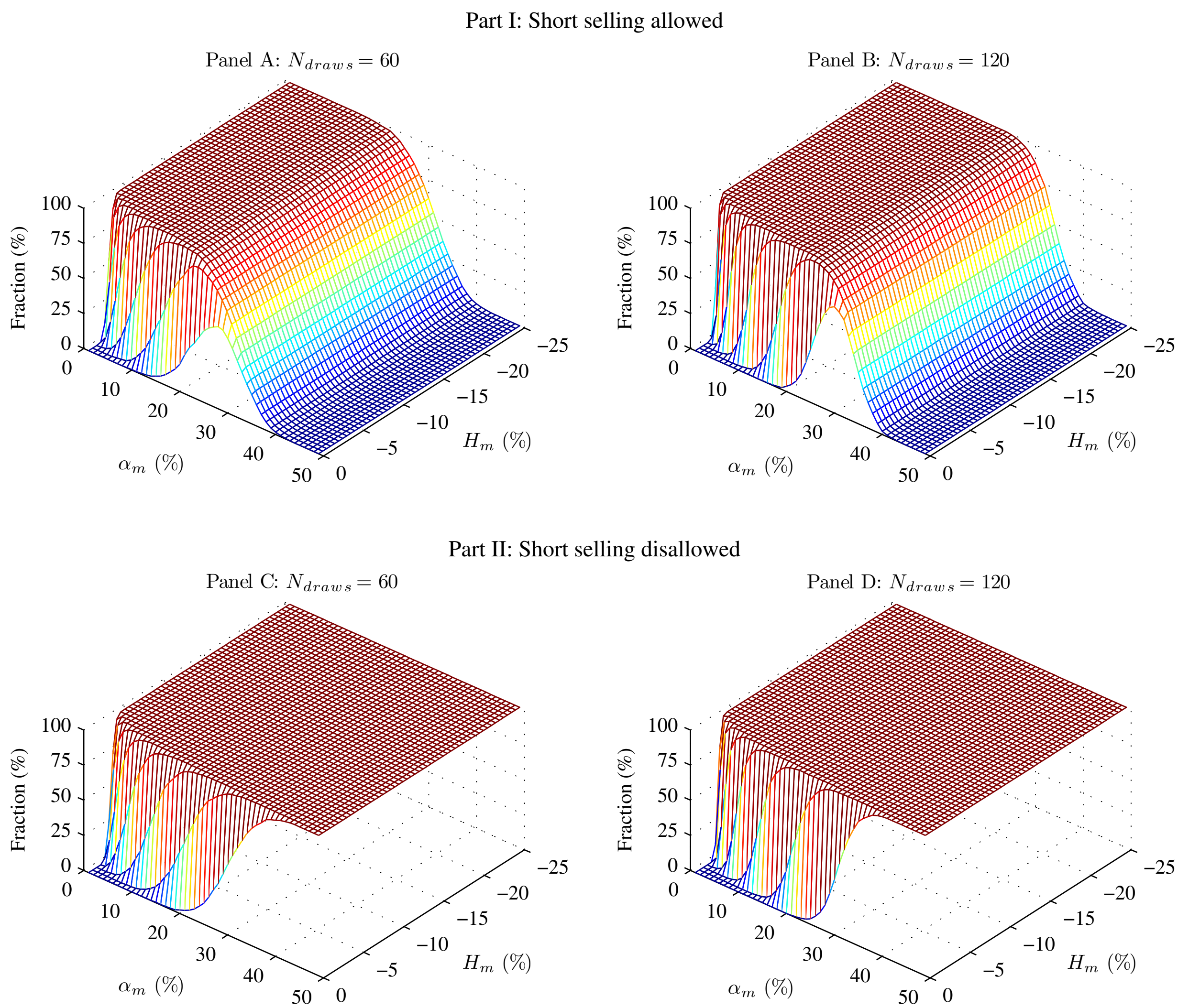


Fig. 3: Average CERs of optimal portfolios within accounts using fixed threshold probabilities and returns as well as simulated data

This figure presents average CERs of optimal portfolios within accounts using fixed threshold probabilities and returns as well as simulated data. The number of draws used to find the estimated optimization inputs, N_{draws}, is either 60 or 120. There are three accounts ($m = 1, 2, 3$). Each panel reports the average CERs of optimal portfolios within accounts 1, 2, and 3 using risk aversion coefficients of, respectively, 4 (solid line), 3 (dashed line), and 1 (dotted line). In panels A, B, E, and F, the thresholds probabilities of accounts 1, 2, and 3 are given by $(\alpha_1, \alpha_2, \alpha_3) = (1\%, 5\%, 10\%)$, whereas their threshold returns $\{H_m\}_{m=1}^{3}$ range from -25% to -5%. In panels C, D, G, and H, the thresholds returns of accounts 1, 2, and 3 are given by $(H_1, H_2, H_3) = (-5\%, -8\%, -10\%)$, whereas their threshold probabilities $\{\alpha_m\}_{m=1}^{3}$ range from 1% to 25%. While short selling is allowed in panels A–D, it is disallowed in panels E–H.

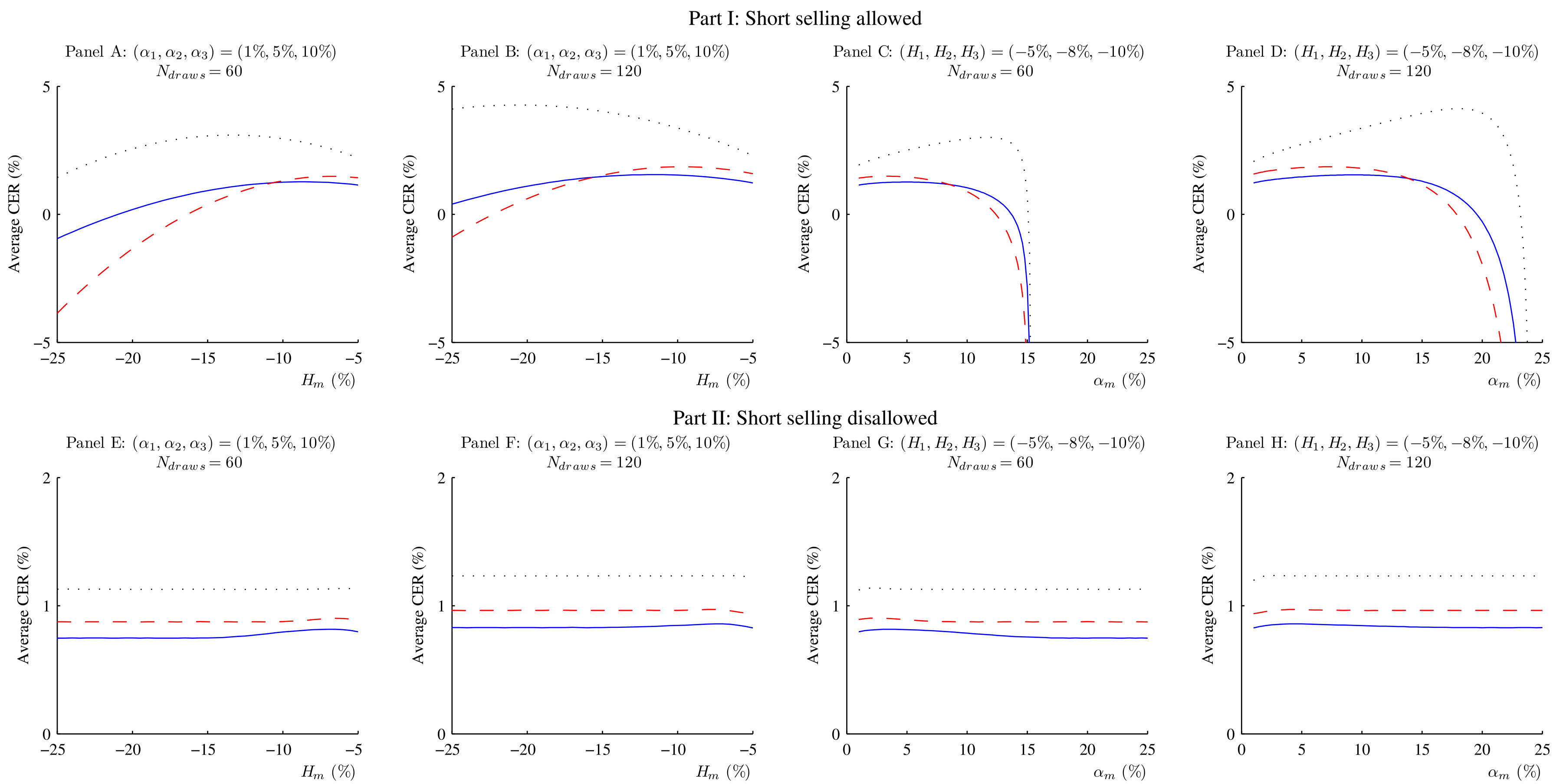


Fig. 4: Box plots of risk aversion coefficients implied by optimal portfolios within accounts using fixed threshold probabilities and returns as well as simulated data

This figure presents box plots of the risk aversion coefficients implied by optimal portfolios within accounts using fixed threshold probabilities and returns as well as simulated data. The number of draws used to find the estimated optimization inputs, N_{draws}, is either 60 or 120. While short selling is allowed in panels A–C, it is disallowed in panels D–F. Panels A and D consider account 1. Similarly, panels B and E consider account 2, whereas panels C and F consider account 3. In panels A–C and D–F, the threshold probabilities (α_m, $m = 1, 2, 3$) and returns (H_m, $m = 1, 2, 3$) are the same as in, respectively, panels A and B of Table 2.

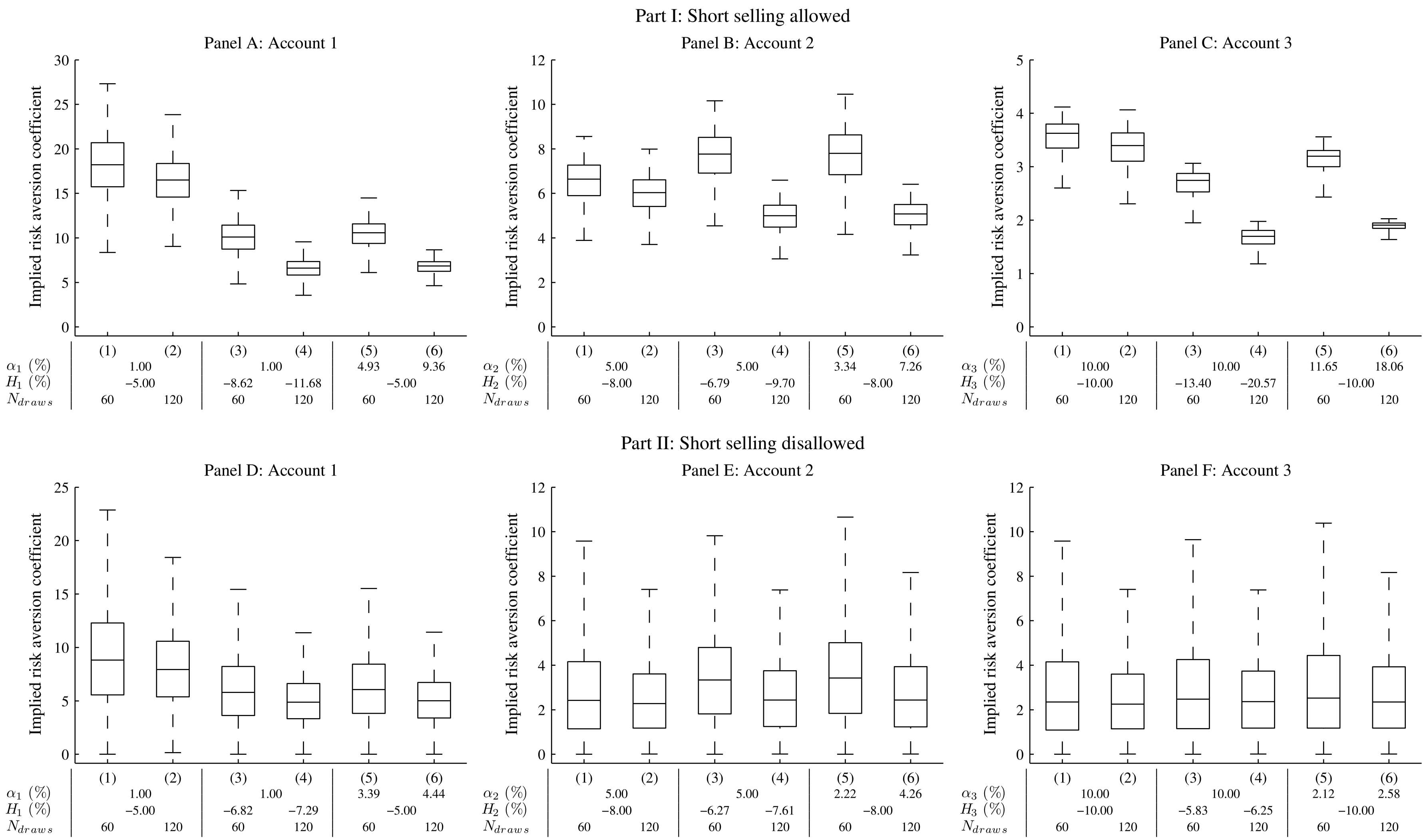


Fig. 5: Average CERs of optimal portfolios within accounts using variable threshold probabilities and returns as well as simulated data

This figure presents average CERs of optimal portfolios within accounts using variable threshold probabilities and returns as well as simulated data. The number of draws used to find the estimated optimization inputs, N_{draws}, is either 60 or 120. There are three accounts ($m = 1, 2, 3$). In determining the average CERs for accounts 1, 2, and 3, we use risk aversion coefficients of, respectively, 4, 3, and 1. The variable threshold probabilities and returns are set so that the optimal portfolio within any given account implies a risk aversion coefficient that does not depend on the values of the estimated optimization inputs. In panels A–F, this coefficient ranges from 1 to 20 and short selling is allowed. In panels G–L, the coefficient ranges from 0 to 20 and short selling is disallowed.

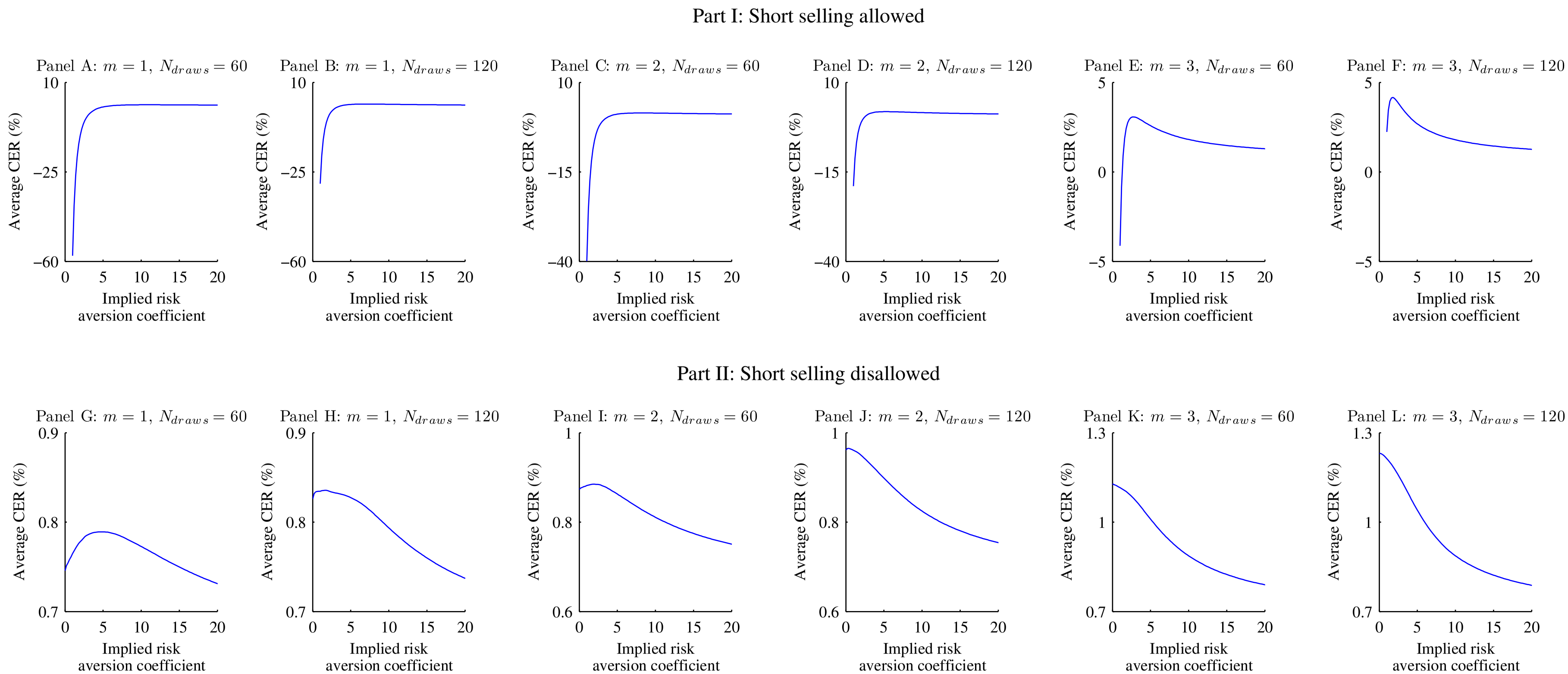
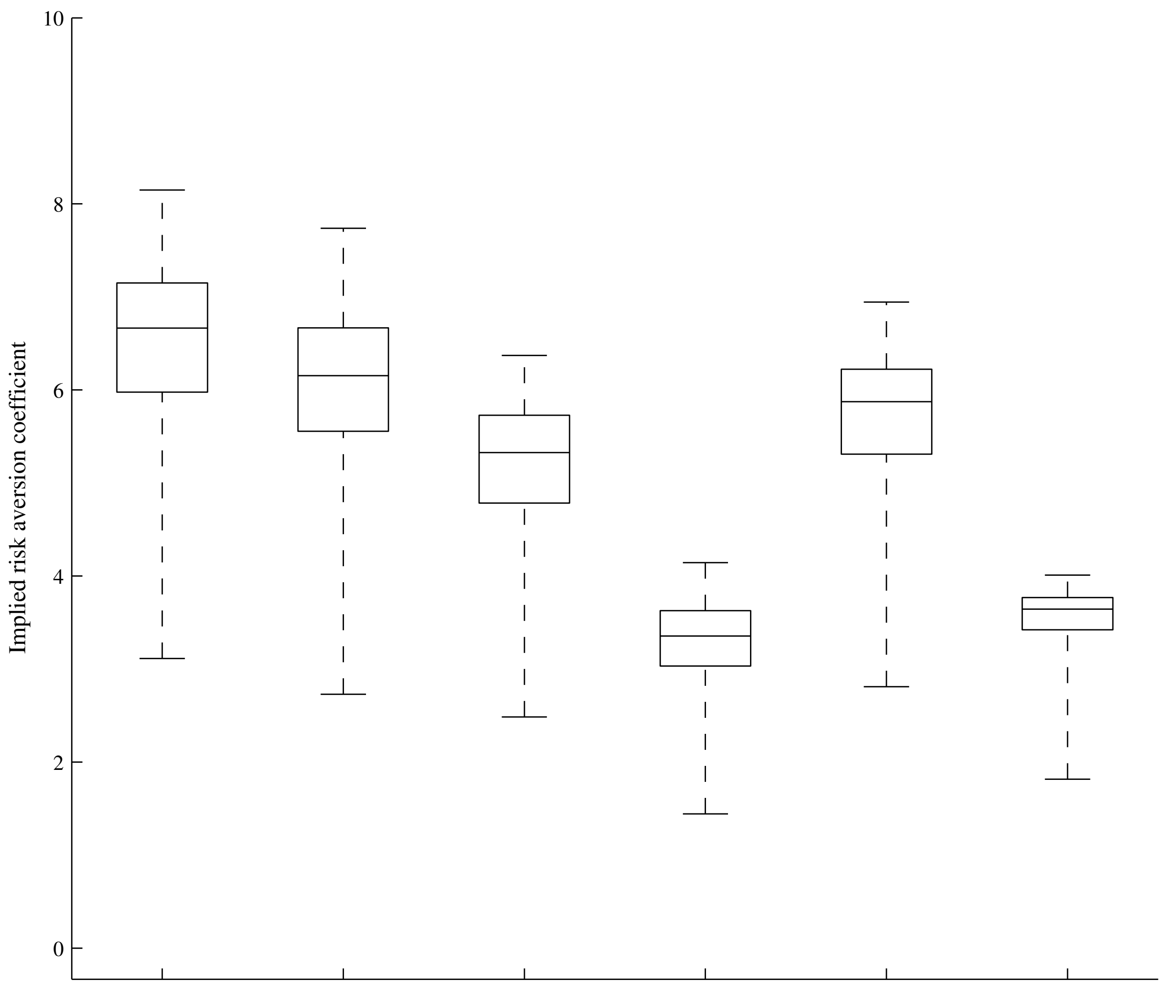


Fig. 6: Box plots of risk aversion coefficients implied by aggregate portfolios using fixed threshold probabilities and returns as well as simulated data

This figure presents box plots of the risk aversion coefficients implied by aggregate portfolios using fixed threshold probabilities and returns as well as simulated data. The number of draws used to find the estimated optimization inputs, N_{draws}, is either 60 or 120. Short selling is allowed. The fractions of wealth in accounts 1, 2, and 3 are, respectively, 40%, 30%, and 30%. Threshold probabilities (α_m, $m = 1, 2, 3$) and returns (H_m, $m = 1, 2, 3$) are the same as in panel A of Table 3.



	(1)	(2)	(3)	(4)	(5)	(6)
α_1 (%)	1.00		1.00		4.93	9.36
α_2 (%)	5.00		5.00		3.34	7.26
α_3 (%)	10.00		10.00		11.65	18.06
H_1 (%)	−5.00		−8.62	−11.68	−5.00	
H_2 (%)	−8.00		−6.79	−9.70	−8.00	
H_3 (%)	−10.00		−13.40	−20.57	−10.00	
N_{draws}	60	120	60	120	60	120

Online appendix: proofs

The following three lemmas are useful in the proofs of our theoretical results.

Lemma 1. *If $\alpha < \alpha^\varepsilon$, then the portfolio with minimum estimated VaR at confidence level $1-\alpha$, denoted by $\boldsymbol{w}_\alpha$, has an estimated VaR at this confidence level of $V^\varepsilon_{1-\alpha} \equiv -H^\varepsilon_\alpha$.*

Proof. Suppose that $\alpha < \alpha^\varepsilon$. Using Eq. (4), portfolio $\boldsymbol{w}_\alpha$ is on the estimated MV frontier. It follows from Eqs. (4) and (8) that $E^\varepsilon[r_{\boldsymbol{w}_\alpha}]$ solves:

$$\min_{E\in\mathbb{R}} \quad z_\alpha\sqrt{1/C^\varepsilon + \frac{(E - A^\varepsilon/C^\varepsilon)^2}{D^\varepsilon/C^\varepsilon}} - E. \tag{33}$$

A first-order condition for $E^\varepsilon[r_{\boldsymbol{w}_\alpha}]$ to solve problem (33) is:

$$\frac{z_\alpha\left(E^\varepsilon[r_{\boldsymbol{w}_\alpha}] - A^\varepsilon/C^\varepsilon\right)/\left(D^\varepsilon/C^\varepsilon\right)}{\sqrt{1/C^\varepsilon + \left(E^\varepsilon[r_{\boldsymbol{w}_\alpha}] - A^\varepsilon/C^\varepsilon\right)^2/\left(D^\varepsilon/C^\varepsilon\right)}} - 1 = 0. \tag{34}$$

It follows from Eq. (34) that:

$$E^\varepsilon[r_{\boldsymbol{w}_\alpha}] = \sqrt{\frac{(D^\varepsilon)^2/\left(C^\varepsilon\right)^3}{z_\alpha^2 - D^\varepsilon/C^\varepsilon}} + A^\varepsilon/C^\varepsilon. \tag{35}$$

Using Eqs. (8) and (35), we have:

$$\sigma^\varepsilon[r_{\boldsymbol{w}_\alpha}] = \sqrt{\frac{z_\alpha^2/C^\varepsilon}{z_\alpha^2 - D^\varepsilon/C^\varepsilon}}. \tag{36}$$

Eqs. (4), (10), (35), and (36) imply that $V^\varepsilon[1-\alpha, r_{\boldsymbol{w}_\alpha}] = \sqrt{\frac{z_\alpha^2 - D^\varepsilon/C^\varepsilon}{C^\varepsilon}} - A^\varepsilon/C = -H^\varepsilon_\alpha$. ■

Lemma 2. *Fix any account $m \in \{1, ..., M\}$. If $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$, then the optimal portfolio within account m, $\boldsymbol{w}^\varepsilon_m$, is on the estimated MV frontier. Furthermore, we have $E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}] > A^\varepsilon/C^\varepsilon$ and $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^\varepsilon_m}] = -H_m$.*

Proof. Fix any account $m \in \{1, ..., M\}$. Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$. First, we show that portfolio $\boldsymbol{w}^\varepsilon_m$ is on the estimated MV frontier. Assume by way of a contradiction that it is not. Then, there exists a portfolio $\boldsymbol{w}$ with $E^\varepsilon[r_{\boldsymbol{w}}] = E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$ and $\sigma^\varepsilon[r_{\boldsymbol{w}}] < \sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$. Let $\boldsymbol{w}^* \equiv \zeta\boldsymbol{w}^\varepsilon_{E_1} + (1-\zeta)\boldsymbol{w}$ where $\zeta > 0$ is arbitrarily small and $E_1 > E^\varepsilon[r_{\boldsymbol{w}}]$. Note that $E^\varepsilon[r_{\boldsymbol{w}^*}] > E^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$ and $\sigma^\varepsilon[r_{\boldsymbol{w}^*}] < \sigma^\varepsilon[r_{\boldsymbol{w}^\varepsilon_m}]$. Hence, it follows from Eq. (4), that $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^*}] < V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^\varepsilon_m}]$.

Inequalities $E^\varepsilon[r_{\boldsymbol{w}^*}] > E^\varepsilon[r_{\boldsymbol{w}_m^\varepsilon}]$ and $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^*}] < V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}_m^\varepsilon}]$ contradict the fact that $\boldsymbol{w}_m^\varepsilon$ is the optimal portfolio within account m. This completes the first part of our proof.

Second, we show that $E^\varepsilon[r_{\boldsymbol{w}_m^\varepsilon}] > A^\varepsilon/C^\varepsilon$. Letting $E \equiv E^\varepsilon[r_{\boldsymbol{w}_m^\varepsilon}]$, Eqs. (4) and (8) imply that:

$$V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}_E^\varepsilon}] = z_{\alpha_m}\sqrt{1/C^\varepsilon + \left(E^\varepsilon[r_{\boldsymbol{w}_E^\varepsilon}] - A^\varepsilon/C^\varepsilon\right)^2 / \left(D^\varepsilon/C^\varepsilon\right)} - E^\varepsilon[r_{\boldsymbol{w}_E^\varepsilon}]. \tag{37}$$

It follows from Eq. (37) that:

$$\frac{\partial V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}_E^\varepsilon}]}{\partial E^\varepsilon[r_{\boldsymbol{w}_E^\varepsilon}]} = \frac{z_{\alpha_m}\left(E^\varepsilon[r_{\boldsymbol{w}_E^\varepsilon}] - A^\varepsilon/C^\varepsilon\right)/\left(D^\varepsilon/C^\varepsilon\right)}{\sqrt{1/C^\varepsilon + \left(E^\varepsilon[r_{\boldsymbol{w}_E^\varepsilon}] - A^\varepsilon/C^\varepsilon\right)^2 / \left(D^\varepsilon/C^\varepsilon\right)}} - 1. \tag{38}$$

Since $z_{\alpha_m} > 0$, Eq. (38) implies that if $E^\varepsilon[r_{\boldsymbol{w}_m^\varepsilon}] \leq A^\varepsilon/C^\varepsilon$, then $\partial V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}_E^\varepsilon}]/\partial E^\varepsilon[r_{\boldsymbol{w}_E^\varepsilon}] < 0$. Hence, we have $E^\varepsilon[r_{\boldsymbol{w}_m^\varepsilon}] > A^\varepsilon/C^\varepsilon$. This completes the second part of our proof.

Third, we show that $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}_m^\varepsilon}] = -H_m$. Eq. (5) implies that $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}_m^\varepsilon}] \leq -H_m$. Assume by way of a contradiction that $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}_m^\varepsilon}] < -H_m$. Let $\boldsymbol{w}^{**} \equiv \delta \boldsymbol{w}_{E_2}^\varepsilon + (1-\delta)\boldsymbol{w}_m^\varepsilon$ where $\delta > 0$ is arbitrarily small and $E_2 > E^\varepsilon[r_{\boldsymbol{w}_m^\varepsilon}]$. Note that $E^\varepsilon[r_{\boldsymbol{w}^{**}}] > E^\varepsilon[r_{\boldsymbol{w}_m^\varepsilon}]$ and $V^\varepsilon[1-\alpha_m, r_{\boldsymbol{w}^{**}}] < -H_m$, which contradict the fact that $\boldsymbol{w}_m^\varepsilon$ is the optimal portfolio within account m. This completes the third part of our proof. ■

Lemma 3. *Fix any $\gamma > 0$ and an objective function $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined by:*

$$f(E^\varepsilon[r_{\boldsymbol{w}}], \sigma^\varepsilon[r_{\boldsymbol{w}}]) = E^\varepsilon[r_{\boldsymbol{w}}] - \frac{\gamma}{2}\left(\sigma^\varepsilon[r_{\boldsymbol{w}}]\right)^2. \tag{39}$$

Letting $E_{\gamma,f}$ denote the estimated expected return of the optimal portfolio associated with γ and f, we have $\frac{D^\varepsilon/C^\varepsilon}{E_{\gamma,f} - A^\varepsilon/C^\varepsilon} = \gamma$.

Proof of Lemma 3. Fix any $\gamma > 0$ and an objective function $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined by Eq. (39). Note that the corresponding optimal portfolio is on the estimated MV frontier. Using Eqs. (8) and (39), $E_{\gamma,f}$ solves:

$$\max_{E \in \mathbb{R}} \quad E - \frac{\gamma}{2}\left[1/C^\varepsilon + \frac{(E - A^\varepsilon/C^\varepsilon)^2}{D^\varepsilon/C^\varepsilon}\right]. \tag{40}$$

A first-order condition for $E_{\gamma,f}$ to solve (40) is $1 - \gamma\frac{E_{\gamma,f} - A^\varepsilon/C^\varepsilon}{D^\varepsilon/C^\varepsilon} = 0$. Hence, $\frac{D^\varepsilon/C^\varepsilon}{E_{\gamma,f} - A^\varepsilon/C^\varepsilon} = \gamma$. ■

Proof of Theorem 1. Fix any account $m \in \{1, ..., M\}$. First, we show (i). Suppose that $\alpha_m \geq \alpha^\varepsilon$. Using the definition of z_{α_m} and (9), we have:

$$0 < z_{\alpha_m} \leq \sqrt{D^\varepsilon/C^\varepsilon}. \tag{41}$$

Fix any level of estimated expected return $E \in \mathbb{R}$. Note that:

$$\frac{\left(E^{\varepsilon}[r_{\boldsymbol{w}_E^{\varepsilon}}] - A^{\varepsilon}/C^{\varepsilon}\right) / \left(D^{\varepsilon}/C^{\varepsilon}\right)}{\sqrt{1/C^{\varepsilon} + \left(E^{\varepsilon}[r_{\boldsymbol{w}_E^{\varepsilon}}] - A^{\varepsilon}/C^{\varepsilon}\right)^2 / \left(D^{\varepsilon}/C^{\varepsilon}\right)}} < \frac{1}{\sqrt{D^{\varepsilon}/C^{\varepsilon}}}. \tag{42}$$

Using Eqs. (38), (41), and (42), we have $\frac{\partial V^{\varepsilon}[1-\alpha_m, r_{w_E^{\varepsilon}}]}{\partial E^{\varepsilon}[r_{w_E^{\varepsilon}}]} < 0$. It follows that the optimal portfolio within account m does not exist.

Suppose now that $\alpha_m < \alpha^{\varepsilon}$ and $H_m > H^{\varepsilon}_{\alpha_m}$. Note that $-H_m < -H^{\varepsilon}_{\alpha_m} = V^{\varepsilon}_{1-\alpha_m}$. Hence, there exists no portfolio $\boldsymbol{w}$ that meets constraint (5). Therefore, the optimal portfolio within account m does not exist. This completes our proof of part (i).

Second, we show part (ii). Suppose that $\alpha_m < \alpha^{\varepsilon}$ and $H_m \leq H^{\varepsilon}_{\alpha_m}$. Lemma 2 and Eq. (8) imply that:

$$E^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}] = A^{\varepsilon}/C^{\varepsilon} + \sqrt{\left(D^{\varepsilon}/C^{\varepsilon}\right)\left[\left(\sigma^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}]\right)^2 - 1/C^{\varepsilon}\right]}. \tag{43}$$

Using Eqs. (4) and (43) along with Lemma 2, we have:

$$z_{\alpha_m}\sigma^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}] - A^{\varepsilon}/C^{\varepsilon} - \sqrt{\left(D^{\varepsilon}/C^{\varepsilon}\right)\left[\left(\sigma^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}]\right)^2 - 1/C^{\varepsilon}\right]} = -H_m. \tag{44}$$

It follows from Eq. (44) that:

$$K_1\left(\sigma^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}]\right)^2 + K_2\sigma^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}] + K_3 = 0, \tag{45}$$

where $K_1 \equiv z^2_{\alpha_m} - D^{\varepsilon}/C^{\varepsilon}$, $K_2 \equiv -2z_{\alpha_m}\left(A^{\varepsilon}/C^{\varepsilon} - H_m\right)$, and $K_3 \equiv \left(A^{\varepsilon}/C^{\varepsilon} - H_m\right)^2 + D^{\varepsilon}/\left(C^{\varepsilon}\right)^2$. Using Eq. (45), we have:

$$\sigma^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}] = \frac{z_{\alpha_m}\left(A^{\varepsilon}/C^{\varepsilon} - H_m\right) \pm \sqrt{\left(D^{\varepsilon}/C^{\varepsilon}\right)\left[\left(A^{\varepsilon}/C^{\varepsilon} - H_m\right)^2 - \left(z^2_{\alpha_m} - D^{\varepsilon}/C^{\varepsilon}\right)/C^{\varepsilon}\right]}}{z^2_{\alpha_m} - D^{\varepsilon}/C^{\varepsilon}}. \tag{46}$$

It follows from Eq. (10) that $H^{\varepsilon}_{\alpha_m} < A^{\varepsilon}/C^{\varepsilon}$. Noting that $H_m \leq H^{\varepsilon}_{\alpha_m} < A^{\varepsilon}/C^{\varepsilon}$, we have $A^{\varepsilon}/C^{\varepsilon} - H_m > 0$. Using the fact that $\alpha_m < \alpha^{\varepsilon}$ and Eq. (9), we obtain $z^2_{\alpha_m} - D^{\varepsilon}/C^{\varepsilon} > 0$. Since $A^{\varepsilon}/C^{\varepsilon} - H_m > 0$, $z^2_{\alpha_m} - D^{\varepsilon}/C^{\varepsilon} > 0$, and $\boldsymbol{w}_m^{\varepsilon}$ solves maximization problem (1) subject to constraints (2) and (5), Eqs. (43) and (46) imply that:

$$\sigma^{\varepsilon}[r_{\boldsymbol{w}_m^{\varepsilon}}] = \frac{z_{\alpha_m}\left(A^{\varepsilon}/C^{\varepsilon} - H_m\right) + \sqrt{\left(D^{\varepsilon}/C^{\varepsilon}\right)\left[\left(A^{\varepsilon}/C^{\varepsilon} - H_m\right)^2 - \left(z^2_{\alpha_m} - D^{\varepsilon}/C^{\varepsilon}\right)/C^{\varepsilon}\right]}}{z^2_{\alpha_m} - D^{\varepsilon}/C^{\varepsilon}}. \tag{47}$$

Eqs. (11)–(13) follow from Lemma 2 along with Eqs. (7), (43), and (47). This completes our proof of part (ii).■

Proof of Corollary 1. Fix any account $m \in \{1, ..., M\}$ with $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$. Eq. (16) follows from Theorem 1 and Lemma 3. ■

Proof of Theorem 2. Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$ for any account $m \in \{1, ..., M\}$. Eqs. (18) and (19) follow from Theorem 1. Using Eqs. (7) and (18), the aggregate portfolio is on the estimated MV frontier. Hence, Eq. (20) follows from Eqs. (8) and (19). ■

Proof of Corollary 2. Suppose that $\alpha_m < \alpha^\varepsilon$ and $H_m \leq H^\varepsilon_{\alpha_m}$ for any $m \in \{1, ..., M\}$. Eq. (23) follows from Theorem 2 and Lemma 3. ■

Proof of Theorem 3. Fix any account $m \in \{1, ..., M\}$ and any constant $\gamma^i_m > 0$. Suppose that $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ satisfy, respectively, Eqs. (25) and (26). Noting that $\gamma^i_m > 0$, Eqs. (9) and (24) imply that $\alpha^{\varepsilon,\gamma^i_m} < \alpha^\varepsilon$. Since $\alpha^{\varepsilon,\gamma^i_m} < \alpha^\varepsilon$ and $\widetilde{\alpha}_m \leq \alpha^{\varepsilon,\gamma^i_m}$, we have $\widetilde{\alpha}_m < \alpha^\varepsilon$.

We claim that $\widetilde{H}_m \leq H^\varepsilon_{\widetilde{\alpha}_m}$. In order to prove this claim, it suffices to show that:

$$\widetilde{H}_m - H^\varepsilon_{\widetilde{\alpha}_m} = 0 \text{ if } z_{\widetilde{\alpha}_m} = \sqrt{[D^\varepsilon + (\gamma^i_m)^2]/C^\varepsilon} \tag{48}$$

and:

$$\left.\frac{\partial(\widetilde{H}_m - H^\varepsilon_{\widetilde{\alpha}_m})}{\partial z_{\widetilde{\alpha}_m}}\right|_{z_{\widetilde{\alpha}_m}=z} \leq 0 \text{ for any } z \geq \sqrt{[D^\varepsilon + (\gamma^i_m)^2]/C^\varepsilon}. \tag{49}$$

Assume that $z_{\widetilde{\alpha}_m} = \sqrt{[D^\varepsilon + (\gamma^i_m)^2]/C^\varepsilon}$. It follows from Eq. (26) that $\widetilde{H}_m = \frac{A^\varepsilon}{C^\varepsilon} - \frac{\gamma^i_m}{C^\varepsilon}$. Using Eq. (10) with $\alpha = \widetilde{\alpha}_m$, we have $H^\varepsilon_{\widetilde{\alpha}_m} = \frac{A^\varepsilon}{C^\varepsilon} - \frac{\gamma^i_m}{C^\varepsilon}$. Hence, Eq. (48) holds. Eqs. (10) and (26) imply that:

$$\left.\frac{\partial(\widetilde{H}_m - H^\varepsilon_{\widetilde{\alpha}_m})}{\partial z_{\widetilde{\alpha}_m}}\right|_{z_{\widetilde{\alpha}_m}=z} = -\sqrt{\frac{1}{C^\varepsilon}\left[1 + \frac{D^\varepsilon}{(\gamma^i_m)^2}\right]} + \sqrt{\frac{1}{C^\varepsilon}\left(\frac{z^2}{z^2 - D^\varepsilon/C^\varepsilon}\right)}. \tag{50}$$

Using Eq. (50), we have:

$$\left.\frac{\partial(\widetilde{H}_m - H^\varepsilon_{\widetilde{\alpha}_m})}{\partial z_{\widetilde{\alpha}_m}}\right|_{z_{\widetilde{\alpha}_m}=\sqrt{[D^\varepsilon+(\gamma^i_m)^2]/C^\varepsilon}} = 0. \tag{51}$$

Note that:

$$\frac{\partial\sqrt{\frac{1}{C^\varepsilon}\left(\frac{z^2}{z^2-D^\varepsilon/C^\varepsilon}\right)}}{\partial z} \leq 0. \tag{52}$$

Eqs. (50)-(52) imply that Eq. (49) holds.

Since $\widetilde{\alpha}_m < \alpha^{\varepsilon}$ and $\widetilde{H}_m \le H^{\varepsilon}_{\widetilde{\alpha}_m}$, part (ii) of Theorem 1 is applicable. Using $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ instead of, respectively, α_m and H_m in Eq. (13), and Eq. (26), the standard deviation of portfolio $\widetilde{\boldsymbol{w}}^{\varepsilon}_m$ is:

$$\widetilde{\sigma}^{\varepsilon}_m = \frac{-\frac{z_{\widetilde{\alpha}_m} D^{\varepsilon}}{\gamma^i_m C^{\varepsilon}} + z^2_{\widetilde{\alpha}_m}\sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}} + \sqrt{\frac{D^{\varepsilon}}{C^{\varepsilon}}\left[\left(\frac{D^{\varepsilon}}{\gamma^i_m C^{\varepsilon}} - z_{\widetilde{\alpha}_m}\sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}}\right)^2 - \frac{z^2_{\widetilde{\alpha}_m} - D^{\varepsilon}/C^{\varepsilon}}{C^{\varepsilon}}\right]}}{z^2_{\widetilde{\alpha}_m} - \frac{D^{\varepsilon}}{C^{\varepsilon}}}. \tag{53}$$

It follows from Eq. (53) and elementary algebra that:

$$\widetilde{\sigma}^{\varepsilon}_m = \frac{-\frac{z_{\widetilde{\alpha}_m} D^{\varepsilon}}{\gamma^i_m C^{\varepsilon}} + z^2_{\widetilde{\alpha}_m}\sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}} + \frac{D^{\varepsilon}}{C^{\varepsilon}}\sqrt{\left[\frac{z_{\widetilde{\alpha}_m}}{\gamma^i_m} - \sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}}\right]^2}}{z^2_{\widetilde{\alpha}_m} - \frac{D^{\varepsilon}}{C^{\varepsilon}}}. \tag{54}$$

Noting that $\widetilde{\alpha}_m \le \alpha^{\varepsilon,\gamma^i_m}$, we have $z_{\widetilde{\alpha}_m} \ge \sqrt{[D^{\varepsilon} + (\gamma^i_m)^2]/C^{\varepsilon}}$. Since $z_{\widetilde{\alpha}_m} \ge \sqrt{[D^{\varepsilon} + (\gamma^i_m)^2]/C^{\varepsilon}}$ and $\gamma^i_m > 0$, we obtain $\frac{z_{\widetilde{\alpha}_m}}{\gamma^i_m} \ge \sqrt{\frac{1}{C^{\varepsilon}} + \frac{D^{\varepsilon}}{(\gamma^i_m)^2 C^{\varepsilon}}}$. Hence, it follows from Eq. (54) that Eq. (29) holds. ■

Proof of Theorem 4. For any account $m \in \{1, ..., M\}$, suppose that $\widetilde{\alpha}_m$ and $\widetilde{H}_m$ satisfy, respectively, Eqs. (25) and (26) for some constant $\gamma^i_m > 0$. Eqs. (30) and (31) follow from, respectively, Eqs. (27) and (28). Using Eqs. (7) and (30), the aggregate portfolio is on the estimated MV frontier. Hence, Eq. (32) follows from Eqs. (8) and (31). ■